# The Subrank of Random Tensors

Jeroen Zuiddam

University of Amsterdam

Joint work with Harm Derksen and Visu Makam.

- We solved a problem in tensor theory about a notion called the subrank of tensors.
- The subrank was introduced by Strassen in 1987 to study fast matrix multiplication algorithms in CS
- and has connections to several problems in math and physics.
- Our result: We determine the subrank for "random tensors"/ "almost all tensors"/ generic tensors
- Improve on previous bounds of Strassen & Bürgisser from 1991

1. Subrank and Applications
2. Tensor Parameters and Their Value on Random Tensors
3. Subrank of Random Tensors
4. Upper bound
5. Lower bound
6. Tensor Space Decomposition

ingredient

# 1. Subrank and Applications

Two characterizations of rank of a matrix $M \in \mathbb{F}^{n \times n}$

**Decomposition into simple matrices**

minimize

$$M = \sum_{i=1}^{r} u_i \otimes v_i$$

"Create matrix from identity"

Equiv:

$$M = A\, I_r\, B$$

**Gaussian elimination**

$$A\, M\, B = I_r$$

maximize

"Create identity from matrix"

Two different notions of rank of a tensor $T \in \mathbb{F}^{n \times n \times n}$

**Tensor rank**

minimize

$$T = \sum_{i=1}^{r} u_i \otimes v_i \otimes w_i$$

Equiv:

$$T = U \otimes V \otimes W \cdot \sum_{i=1}^{r} e_i \otimes e_i \otimes e_i$$

$R(T)$

## Applications

- Matrix multiplication
- Circuit complexity [Raz]

**Subrank**

maximize

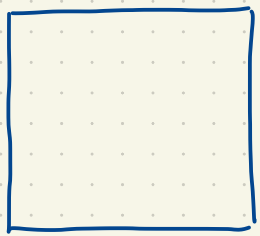
$$\sum_{i=1}^{s} e_i \otimes e_i \otimes e_i = U \otimes V \otimes W \cdot T$$

$Q(T)$

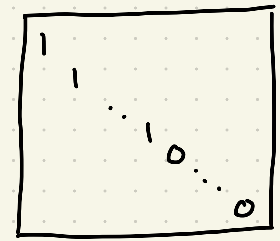
- Matrix Multiplication
- Additive Combinatorics

## Matrix rank

$M$ =

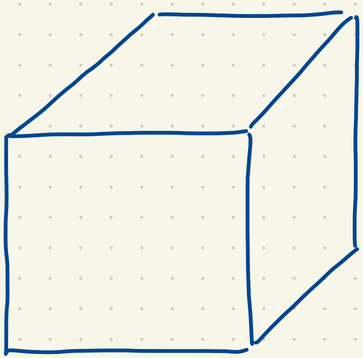


linear combinations of rows and columns

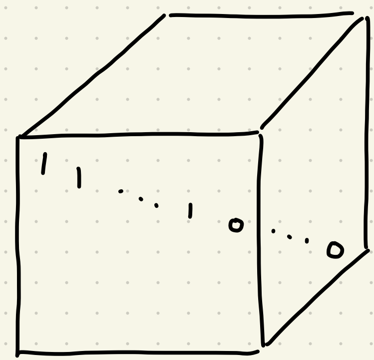


## Subrank

$T$ =



linear combinations of slices in all three directions



# Applications of Subrank

- ## Complexity Theory

$T \in \mathbb{F}^{n \times n \times n}$ $\rightsquigarrow$ bilinear map $T : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$

$Q(T)$ $\rightsquigarrow$ number of independent scalar multiplications that can be reduced to $T$

- ## Quantum Information

$T \in \mathbb{C}^{n \times n \times n}$ $\rightsquigarrow$ Tripartite quantum state

$Q(T)$ $\rightsquigarrow$ largest "GHZ" state obtainable from $T$ by SLOCC

- ## Combinatorics

$H \subseteq [n] \times [n] \times [n]$ hypergraph, independence number $\alpha(H) \leq Q(T)$ for any $T$ that "fits" $H$. E.g. cap sets, sunflowers, corners, ...

## 2. Tensor Parameters and Their Value on Random Tensors

$T \in \mathbb{F}^{n \times n \times n}$

$0 \le Q(T) \le SR(T) \le n \le R(T) \le n^2$

$AR(T)$

$GR(T)$

$R^G(T)$

## Slice rank

$$T = \sum_{i=1}^{a} \sum_{j} u_i \otimes v_{ij} \otimes w_{ij} + \sum_{i=1}^{b} \sum_{j} u'_{ij} \otimes v'_i \otimes w'_{ij}$$

$$+ \sum_{i=1}^{c} \sum_{j} u''_{ij} \otimes v''_{ij} \otimes w''_i$$

minimize $a+b+c$

$SR(T)$

## Geometric rank

$$\operatorname{codim} \{ (u,v) \in \mathbb{F}^n \times \mathbb{F}^n : \forall w \;\; T(u,v,w) = 0 \}$$

$GR(T)$

$T \in \mathbb{F}^{n \times n \times n}$

Generally: $\quad 0 \le Q(T) \le \begin{matrix} SR(T) \\ AR(T) \\ GR(T) \\ R^G(T) \end{matrix} \le n \le R(T) \le n^2$

On random tensors $T$: $\quad Q(T)$: ? $\quad SR(T), AR(T), GR(T), R^G(T)$: $\approx n$ $\quad R(T)$: $\approx n^2$

## 3. Subrank of Random Tensors

**Theorem** For almost all $T \in \mathbb{F}^{n \times n \times n}$ we have $Q(T) = \theta(\sqrt{n})$

**Remarks:**

- "Almost all" = "random" = generic
- that is, there is a nonempty Zarisk-open $\mathcal{U} \subseteq \mathbb{F}^{n \times n \times n}$ such that for all $T \in \mathcal{U}$ we have $Q(T) = \theta(\sqrt{n})$
- Very precise bounds: $\sqrt{3n-2} - 5 \leq Q(T) \leq \sqrt{3n-2}$
- Previously: $Q(T) \leq n^{2/3 + o(1)}$
- Also for higher-order tensors
- Application: Subrank is not additive under direct sum.

## Upper bound

$Q(n) :=$ subrank of a generic tensor in $\mathbb{F}^{n \times n \times n}$

To prove: $Q(n) \leq \sqrt{3n-2}$

$C_r := \{$ tensors in $\mathbb{F}^{n \times n \times n}$ with subrank $\geq r \}$

<u>Lemma 1</u> $Q(n) =$ largest $r$ such that $\dim C_r = \underbrace{\dim \mathbb{F}^{n \times n \times n}}_{n^3}$.

<u>Lemma 2</u> $\dim C_r \leq n^3 - r(r^2 - 3n + 2)$

Let $t = Q(n)$

Then $n^3 = \dim C_t \leq n^3 - t(t^2 - 3n + 2)$.

Then $t^2 - 3n + 2 \leq 0$

So $t \leq \sqrt{3n-2}$

$C_r := \{ \text{tensors in } \mathbb{F}^{n\times n\times n} \text{ with subrank} \geq r \}$

<u>Lemma 2</u> $\dim C_r \leq n^3 - r(r^2 - 3n + 2)$

<u>Proof idea</u>

- Non-injective parametrization of $C_r$
- Compute dimension of parameter space
- Subtract dimension of "over-count" (fiber dimension)

$X_r = \{ \text{tensors in } \mathbb{F}^{n\times n\times n} \text{ with } [r]\times[r]\times[r] \text{ subtensor arbitrary diag.} \}$

$$\begin{aligned} \Psi_r : \ & GL_n \times GL_n \times GL_n \times X_r \to \mathbb{F}^{n\times n\times n} \\ & (A, B, C, T) \longmapsto (A\otimes B\otimes C)T \end{aligned}$$

has image $C_r$

$\square$

## Lower bound

$$X_r = \{ \text{tensors in } \mathbb{F}^{n\times n\times n} \text{ with } [r]\times[r]\times[r] \text{ subtensor arbitrary diag.} \}$$

$$\begin{aligned} \Psi_r : GL_n \times GL_n \times GL_n \times X_r &\to \mathbb{F}^{n\times n\times n} \\ (A, B, C, T) &\mapsto (A\otimes B\otimes C)T \end{aligned} \quad \text{has image } C_r$$

## Proof idea

- Find condition that imply image of $\Psi_r$ has full dimension
- Use notion of differential $d\Psi_r$

$$(d\Psi_r)_{(g_1,g_2,g_3,T)} : \mathrm{Mat}_{n\times n} \times \mathrm{Mat}_{n\times n} \times \mathrm{Mat}_{n\times n} \times Y_r \to \mathbb{F}^{n\times n\times n}$$

$$(A, B, C, T) \mapsto ((A\otimes g_2\otimes g_3) + (g_1\otimes B\otimes g_3) + (g_1\otimes g_2\otimes C))T$$
$$+ (g_1\otimes g_2\otimes g_3)S.$$

# Tensor space decompositions

Goal: write tensor space $\mathbb{F}^{n\times n\times n}$ as a sum of tensor subspaces, as efficiently as possible such that each subspace has the form of an $n\times n$ matrix subspace tensored with $\mathbb{F}^n$

$X \subseteq \mathrm{Mat}_{n\times n} = \mathbb{F}^n \otimes \mathbb{F}^n$

$X[1] = \mathbb{F}^n \otimes X \subseteq \mathbb{F}^{n\times n\times n}$

$X[2] =$

$X[3] = X \otimes \mathbb{F}^n$

**Theorem** there are subspaces $X_i \subseteq \mathrm{Mat}_{3,3}$ of dim 3 each, such that

$$\mathbb{F}^{3\times 3\times 3} = X_1[1] + X_2[2] + X_3[3].$$

**Theorem** There are subspaces $X_i \subseteq \mathrm{Mat}_{3,3}$ of dim 3 each, such that

$$\mathbb{F}^{3\times 3\times 3} = X_1[1] + X_2[2] + X_3[3].$$

Note: dimensions left and right are equal.

**Remark** Not possible with matrices: there are no subspaces $X_i \subseteq \mathbb{F}^n$ of dimension $n/2$ each such that $\mathbb{F}^{n\times n} = X_1[1] + X_2[2]$

**Theorem** There are subspaces $X_i \subseteq (\mathbb{F}^n)^{\otimes n-1}$ of dim $n^{n-2}$ each, such that

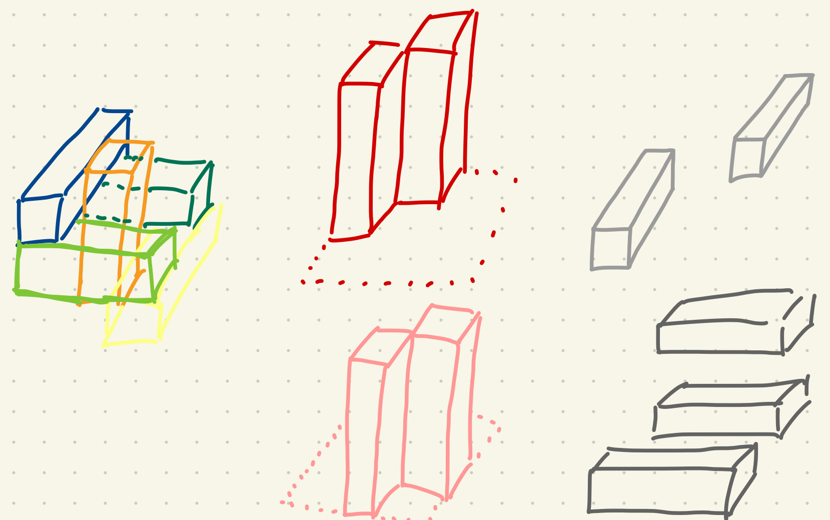
$$(\mathbb{F}^n)^{\otimes n} = X_1[1] + X_2[2] + \cdots + X_n[n].$$

Again: dimensions match.

**Theorem** there are subspaces $X_i \subseteq \mathrm{Mat}_{3,3}$ of dim 3 each, such that

$$\mathbb{F}^{3\times3\times3} = X_1[1] + X_2[2] + X_3[3].$$

| $X_1$ | $X_2$ | $X_3$ |
|---|---|---|
| $\begin{matrix}1&0&0\\0&0&0\\0&0&0\end{matrix}$ | $\begin{matrix}0&0&0\\0&0&1\\0&0&0\end{matrix}$ | $\begin{matrix}0&0&0\\0&1&0\\0&0&0\end{matrix}$ |
| $\begin{matrix}0&0&0\\0&0&0\\0&0&1\end{matrix}$ | $\begin{matrix}0&0&0\\1&0&0\\0&0&0\end{matrix}$ | $\begin{matrix}0&1&0\\1&0&0\\0&0&0\end{matrix}$ |
| $\begin{matrix}0&0&1\\0&0&0\\1&0&0\end{matrix}$ | $\begin{matrix}0&0&1\\0&0&0\\1&0&1\end{matrix}$ | $\begin{matrix}0&0&0\\0&0&1\\0&1&0\end{matrix}$ |



## Application: Subrank is not additive under direct sum

**Theorem** There are tensors $S, T \in \mathbb{F}^{n \times n \times n}$ such that $Q(S), Q(T) \le \sqrt{3n-2}$ while $Q(S \oplus T) \geq n$.

**Proof idea**

- Let $T$ be "random".
- Let $S = I_n - T$. Then $S$ is "random".
- Then $Q(S), Q(T) \le \sqrt{3n-2}$ by our theorem.
- On the other hand, $Q(S \oplus T) \geq Q(S+T) = Q(I_n) = n$. $\square$

# Selected Open Problems

1. Our upper bound $Q(T) \leq \lfloor \sqrt{3n-2} \rfloor$ for generic $T \in \mathbb{F}^{n \times n \times n}$ is tight for $n \leq 100$. Is this always true?

2. Determine all possible tensor space decompositions

3. What is the largest gap between $Q(S \oplus T)$ and $Q(S) + Q(T)$?