Community Detection in Sparse Random Hypergraphs

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Random Tensors and Related Topics
CIRM

Joint work with Soumik Pal (Univeristy of Washington)
and Ludovic Stephan (EPFL)

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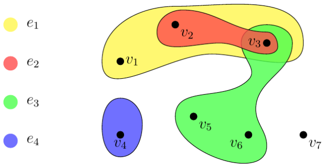

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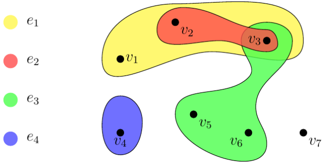

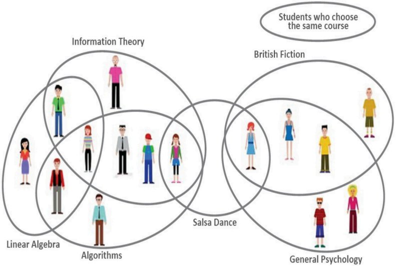

Ravindran '15

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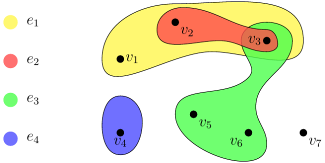

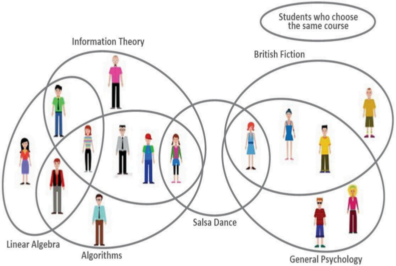

Ravindran '15

- co-authorship network
- chat group in social network
- Protein interaction network

Higher-order network

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SIAM NEWS **BLOG**

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GRAPH THEORY

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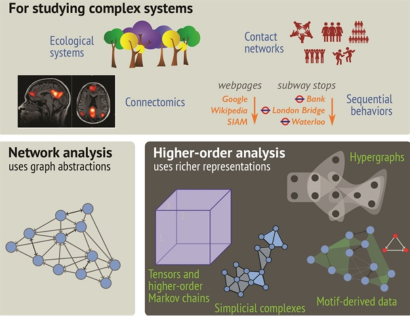

sinews.siam.org/Details-Page/higher-order-network-analysis-takes-off-fueled-by-old-ideas-and-new-data
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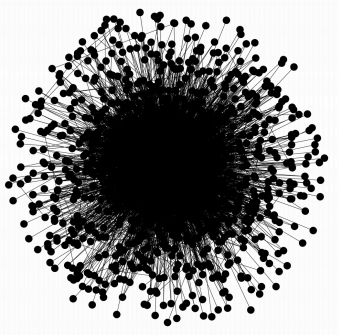
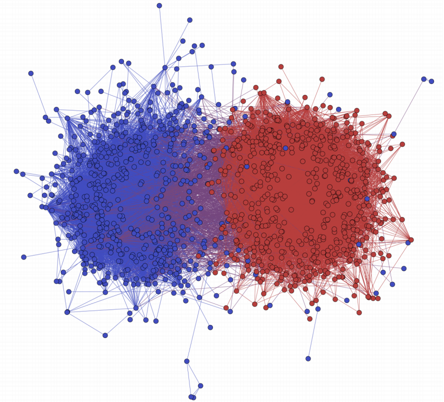
Community detection





Political blogs data from Adamic-Glance '05. Figure from Abbe '18

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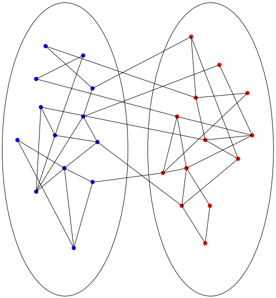

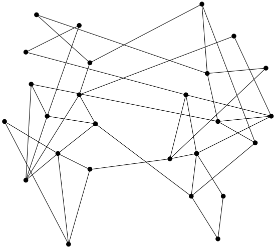

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  Feige–Ofek '05, Lei–Rinaldo '13, Le–Levina–Vershynin '16, Benaych Georges–Bordenave–Knowles '17, Latala–van Handel–Youssef '17, Alt–Ducatez–Knowles '19, Tikhomirov–Youssef '19

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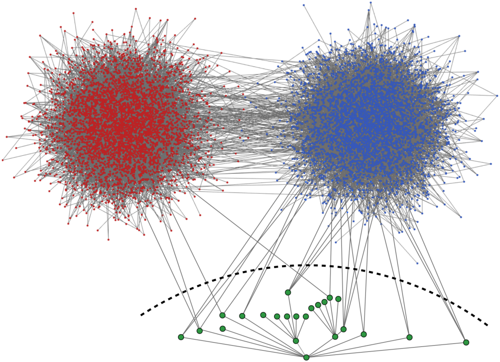
Decelle-Krzakala-Moore-Zdeborová '11, Mossel-Neeman-Sly '12, '14, Massoulié '14, Bordenave-Lelarge-Massoulié '15.
Rich literature on SBMs in more general cases and different settings: survey by Abbe '18.

Bounded expected degrees



Abbe et al. '18, $a = 2.2$, $b = 0.06$, $n = 100000$, apply spectral method directly on A

When $p = \frac{a}{n}, q = \frac{b}{n}$, top eigenvectors are localized on high degree vertices.

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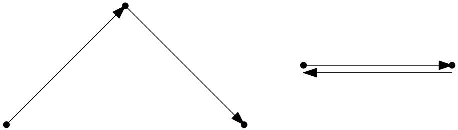

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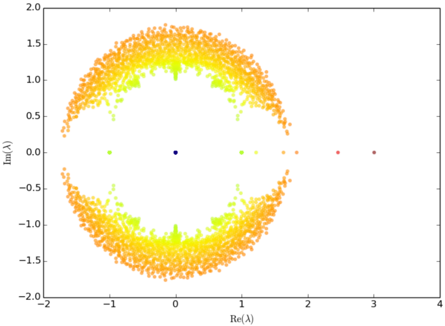

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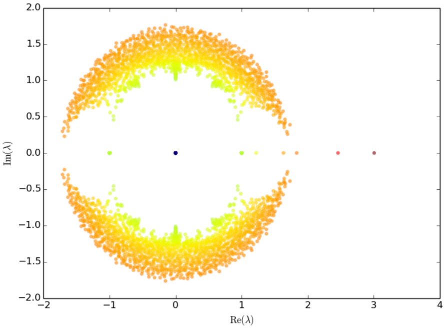

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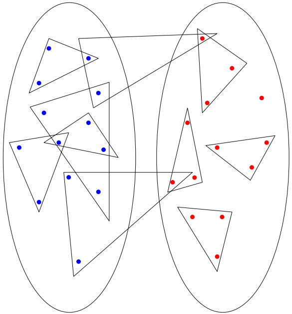

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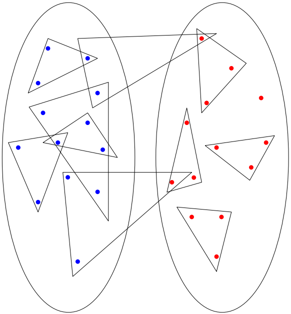

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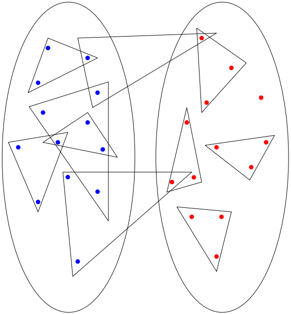

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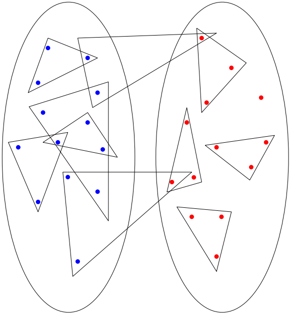

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when expected degree (expected number of hyperedges containing a vertex) $d \to \infty$.

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- (Provable) spectral method in the bounded expected degree regime?

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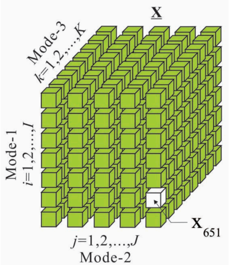

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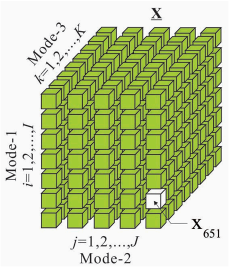

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Most tensor problems are NP-hard (Hillar-Lim '13): rank, spectral norm, best low-rank approximation,...

Tucker decomposition: Ghoshdastidar-Dukkipat '17, Ke-Shi-Xia '20 for $d = \omega(\log n)$.

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[Pal-Z. '21]: spectral method on a matrix counting the self-avoiding walk of length $O(\log n)$ for HSBM achieves the conjectured threshold in Angelini et al '15, generalization of Massoulié '14.

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What about the non-backtracking operator?

Adjacency matrix

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What about the non-backtracking operator?
[Stephan, Z. '22]: Very efficient!

Non-backtracking operator for hypergraphs

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For a given hypergraph $G = (V, H)$, let $\vec{H}$ be the *oriented hyperedge* in G such that

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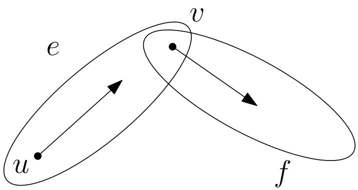

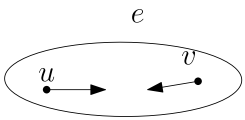

Storm '06: Zeta function of hypergraphs.

Generate an HSBM from a probability tensor

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for any $e = \{v_1, \dots, v_q\}$, where

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- Assume each vertex has the same expected degree d.

Generalized Kesten-Stigum threshold

The nonzero eigenvalues of $\mathbb{E}A$ are given by

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The generalized Kesten-Stigum threshold conjectured in Angelini et al. '15.

Spectrum of B

Theorem (Stephan-Z., '22)

Let G be a hypergraph generated according to the HSBM with m hyperedges, and B be its non-backtracking matrix and $|\lambda_1(B)| \geq |\lambda_2(B)| \geq \cdots \geq |\lambda_{qm}(B)|$. Then with high probability:

1. *For any $i \in [r_0]$,*
$$\lambda_i(B) = (q-1)\mu_i + o(1).$$
2. *For all $r_0 < i \leq qm$,*
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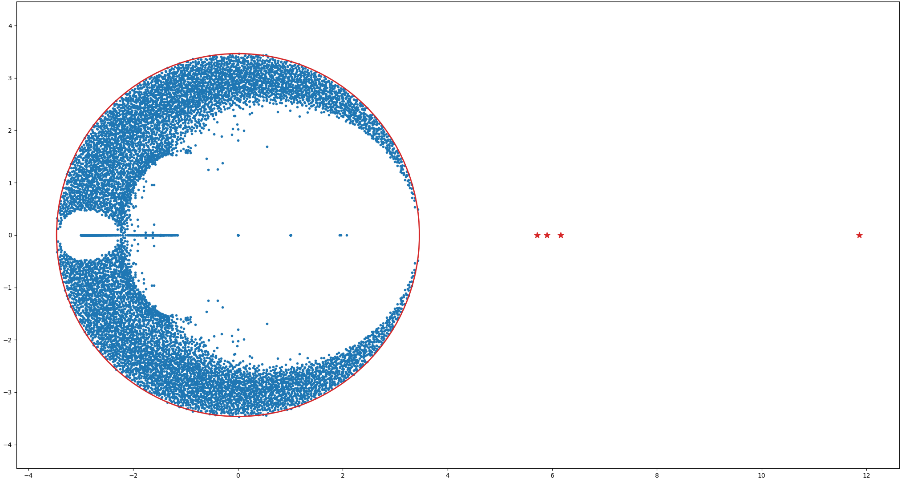
- Informative eigenvalues of $\mathbb{E}A$ above the Kesten-Stigum threshold can be seen in the spectrum of B outside the disk of radius $\sqrt{(q-1)d}$.
- Other eigenvalues of B are confined in the disk.

Spectrum of B



$n = 6000$, $q = r = 4$. The parameters c_{in} and c_{out} have been chosen so that $d = 4$ and $\mu_2 = 2$. The single eigenvalue is close to $(q-1)d = 12$ and the three eigenvalues are near $(q-1)\mu_2 = 6$.

Dimension reduction

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Define the $2n \times 2n$ matrix $\tilde{B}$ as

$$\tilde{B} = \begin{pmatrix} 0 & (D-I) \\ -(q-1)I & A-(q-2)I \end{pmatrix},$$

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The following Ihara-Bass formula holds:

$$\begin{aligned}\det(B - zI) =& (z-1)^{(q-1)|H|-n}(z+(q-1))^{|H|-n} \\ & \cdot \det\left(z^2 + (q-2)z - zA + (q-1)(D-I)\right).\end{aligned}$$

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$q = 2$: Bass '92. Storm '06 for regular hypergraphs, stated in Angelini et al. '15.

Eigenvector overlaps

Theorem (Stephan-Z., '22)

For $i \in [r_0]$, let $\tilde{u}_i$ be the last n entries of the i-th eigenvector of $\tilde{B}$, normalized so that $\|\tilde{u}_i\| = 1$. Then with high probability, there exists a unit eigenvector $\tilde{\phi}_i$ of $\mathbb{E}A$ associated to λ_i such that

$$\langle \tilde{u}_i, \tilde{\phi}_i \rangle = \sqrt{\frac{1-\tau_i}{1+\frac{q-2}{(q-1)\mu_i}}} + o(1) \quad \text{where } \tau_i = \frac{d}{(q-1)\mu_i^2}.$$

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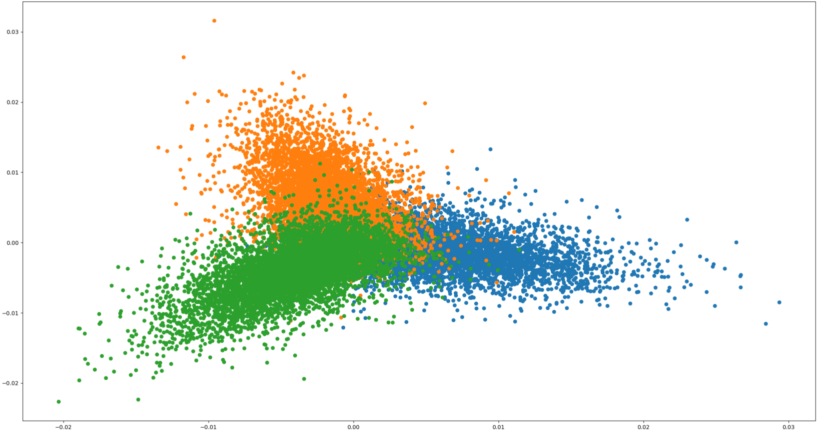
When $r = 2$, and

$$p_{i_1,\dots,i_q} = \begin{cases} c_{\mathrm{in}} & \text{if } \sigma(i_1) = \cdots = \sigma(i_q) \\ c_{\mathrm{out}} & \text{otherwise} \end{cases},$$

rounding the entries $\tilde{u}_2$ to ± 1 gives a correlated detection.

More than 2 blocks



Scatter plot of the second and third eigenvector of $\tilde{B}$ under the symmetric HSBM with $q = 4$, $r = 3$ and $n = 20000$. The parameters c_{in} and c_{out} have been chosen so that $d = 4$ and $\mu_2 = 2$. The colors correspond to the actual label of each vertex.

vertices	1	2	$\cdots$	n
$\tilde{u}_2$	x_1	x_2	$\cdots$	x_n
$\tilde{u}_3$	y_1	y_2	$\cdots$	y_n

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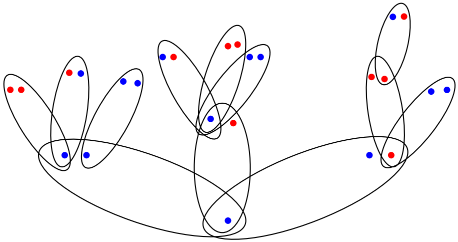

- Start from a *root* ρ with a given spin $\sigma(\rho)$;
- Generate $k = \text{Poi}(d)$ hyperedges intersecting only at ρ, yielding $k(q-1)$ *children*;
- For each hyperedge, fix an ordering of the $(q-1)$ associated children $v = (v_1, \ldots, v_{q-1})$. Assign a type to each $(q-1)$-tuple randomly such that

$$\mathbb{P}\left(\underline{\sigma}(v) = \underline{j}\right) = \frac{1}{d} \cdot p_{\sigma(\rho),\underline{j}} \cdot \prod_{\ell \in \underline{j}} \pi_\ell .$$

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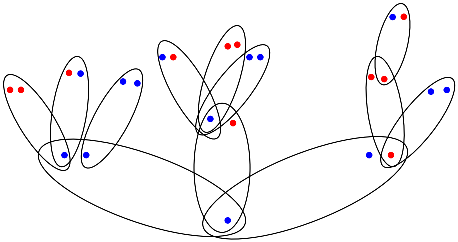

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[Pal-Z. '21]: considered 2-type Galton-Watson hypertrees.

Moment method and a bipartite representation

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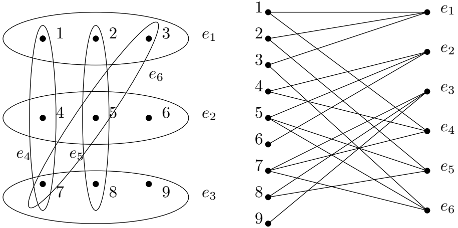

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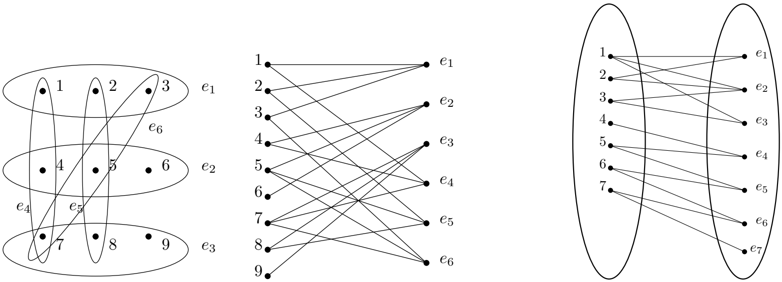

A closed non-backtracking walk: $(1, e_1, 2, e_2, 1, e_3, 3, e_2, 1)$.

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Thank You!