Trisections in Colored Tensor Models

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Abstract

We give a procedure to construct trisection for closed manifolds generated by colored tensor models without restrictions on the number of simplices in the triangulation, therefore generalising previous works of crystallisations [Casali, Cristofori 2019] and of PL-manifolds [Bell, Hass, Rubinstein, Tillmann 2017].

Introduction

Tensor models are a random geometric approach to quantum gravity, which takes the path integral approach with an interpretation that summing over all geometries and topologies with suitable weights should give classical geometry of our universe. $d+1$-colored tensor models with d-indices, in particular, are shown to represent fluctuating d-dimensional piecewise-linear (PL) pseudo-manifolds[1] via their perturbative expansion in Feynman graphs encoding topological spaces. It is then essential to understand the topological and geometrical structure of the PL pseudo-manifolds that colored tensor models generate in order to understand better their path integral formulation.

Organised by the Gurau degree (ω), colored tensor models admit a $1/N$ expansion of the partition function ($\sim \sum_{\mathcal{G}} N^{-\omega(\mathcal{G})}$) with a resummable leading order ($\omega = 0$), called melon graphs, exhibiting critical behavior and a continuum limit. Melons are a subclass of spheres and in the continuum limit are shown to behave like branched polymers (tree) with Hausdorff dimension 2 and the spectral dimension $4/3$. Reflecting and motivated by quantum gravity, we dream of a possibility of finding a new parameter for colored tensor model to classify the graphs in a new large N limit, which may then give some new critical behavior. The Gurau degree arises naturally in the construction of Heegaard splitting of three-dimensional PL manifolds generated by colored tensor models with 3-indices. In four dimensions, there exists analogous topological concept, trisections, introduced by Gay and Kirby in 2012. Trisections are a novel tool to describe $4-$manifolds by revealing the nested structure of lower dimensional submanifolds. In particular, the trisection genus of a $4-$manifold is a topological invariant. In this work, therefore, we formulate trisections in colored tensor models with 4-indices.

Tensor models

Consider combinatorially nonlocal 0-dimensional field theories of size N random tensors with d-indices $T : \mathbb{Z}_N^d \to \mathbb{C}$ (to make it an orientable manifold), where the action for tensors with 4-indices is given by $S[T,\bar{T}] = N^{4/2}\Big(\sum_{c=0}^{5} \sum_{a_i \in \mathbb{Z}_N} T^{c}_{a_1a_2a_3a_4} \bar{T}^{c}_{a_1a_2a_3a_4} + \lambda \sum_{a_{ij}\in\mathbb{Z}_N} T^0_{a_{01}a_{02}a_{03}a_{04}} T^1_{a_{10}a_{14}a_{13}a_{12}} T^2_{a_{21}a_{20}a_{24}a_{23}} T^3_{a_{32}a_{31}a_{30}a_{34}} T^4_{a_{43}a_{42}a_{41}a_{40}}$ $+\bar{\lambda} \sum_{a_{ij}\subset\mathbb{Z}_N} \bar{T}^0_{a_{01}a_{02}a_{03}a_{04}} \bar{T}^1_{a_{10}a_{14}a_{13}a_{12}} \bar{T}^2_{a_{21}a_{20}a_{24}a_{23}} \bar{T}^3_{a_{32}a_{31}a_{30}a_{34}} \bar{T}^4_{a_{43}a_{42}a_{41}a_{40}} \Big)$, with $a_{ij} = a_{ji}$ and the probability measure is given by $d\mu = \int \mathcal{N} \prod_{c,a_i} dT^{i}_{a_1a_2a_3a_4} d\bar{T}^{i}_{a_1a_2a_3a_4} \; e^{-S[T,\bar{T}]}$. The tensors transform under the symmetry of $U(N)^4$: $T_{a_1a_2a_3a_4} \to T'_{a_1a_2a_3a_4} = T_{lmnp} U^l_{a_1} U^m_{a_2} U^n_{a_3} U^p_{a_4}$ and $\bar{T}_{a_1a_2a_3a_4} \to \bar{T}'_{a_1a_2a_3a_4} = \bar{T}_{lmnp} (U^\dagger)^l_{a_1} (U^\dagger)^m_{a_2} (U^\dagger)^n_{a_3} (U^\dagger)^p_{a_4}$

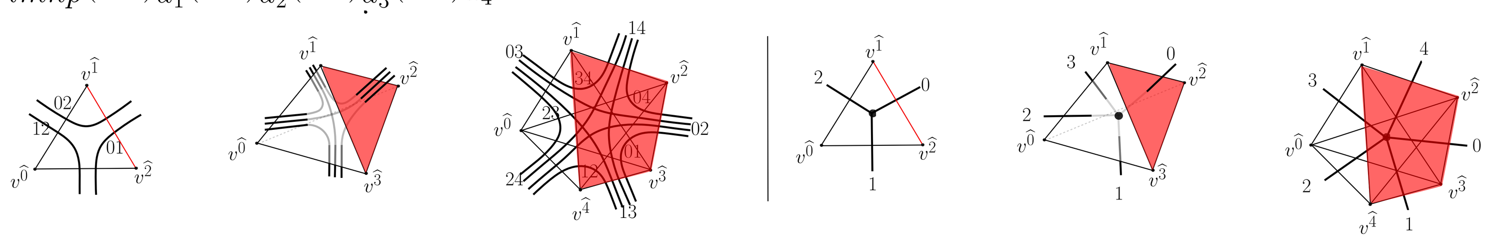
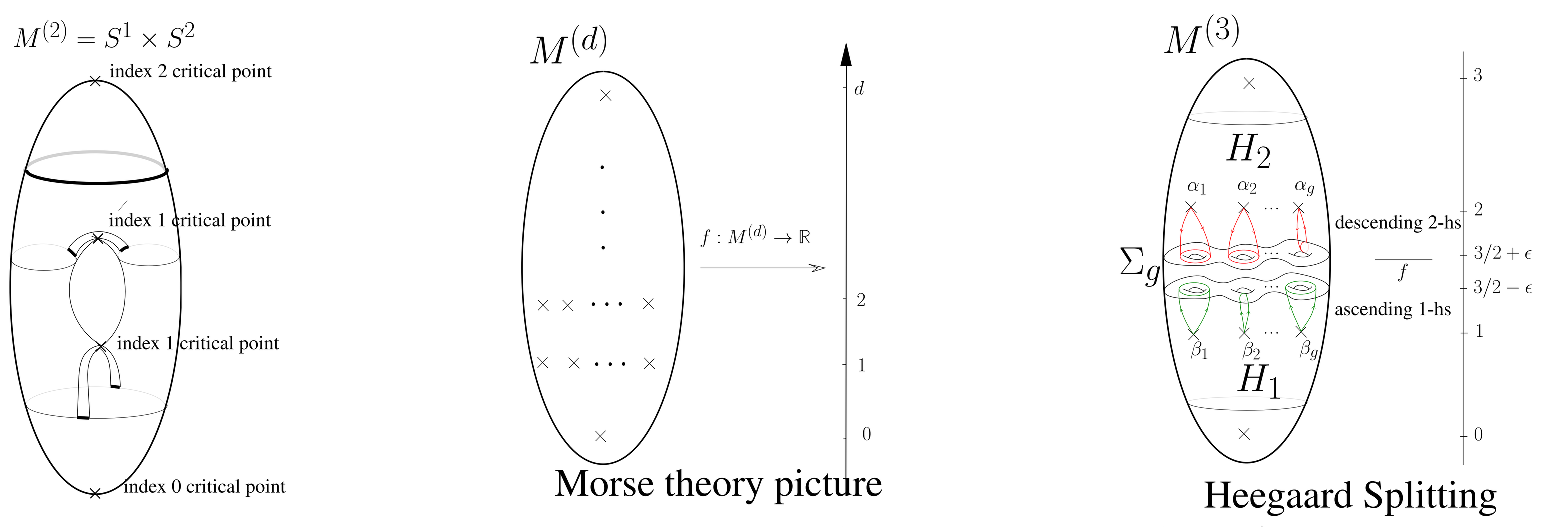


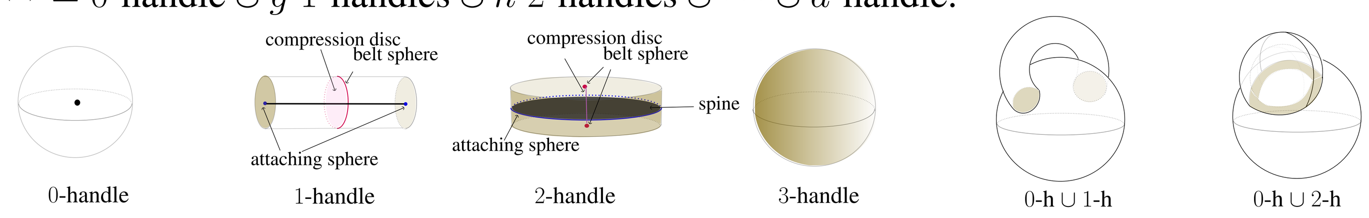
Figure 1: *d-simplices in $d = 2, 3, 4$ dimensions, where we embedded $d+1$-colored graphs. Stranded representation (left) and color representation (right). $(d+1)$-colored graphs (also, called graph encoding manifolds) are dual to simplicial triangulations of PL d-dimensional pseudo-manifolds.*

Handle decomposition (in general d-dimensions) and Heegaard splittings (in 3-dimensions) in TOP category.

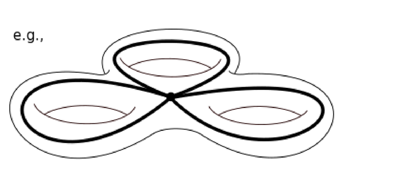


An i-**handle** can be thought of as a thickening of D^i, and is therefore $\mathrm{D}^i \times \mathrm{D}^{d-i}$ glued along $\mathrm{S}^{i-1} \times \mathrm{D}^{d-i}$. In Morse theory, passing through an index i critical point means attaching an i-handle, i.e., topology changes when passing an i-handle. Morse theory picture naturally provides us with the **handle decomposition** of a closed d-manifold: $M^{(d)} =$ 0-handle $\cup\ g$ 1-handles $\cup\ h$ 2-handles $\cup \cdots \cup$ d-handle.

e.g., in 3-dimensions,

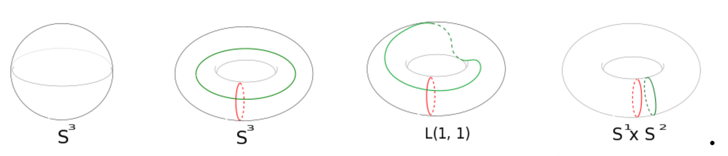


A special case is when $H^{(d)} =$ 0-handle $\cup\ g$ 1-handles (therefore with a boundary), then $H^{(d)}$ is called **1-handlebody** with its genus defined as g. In a 1-handlebody, there is a **spine** (i.e., a graph, oe equivalently a set of vertices and edges) where the manifold collapses onto.

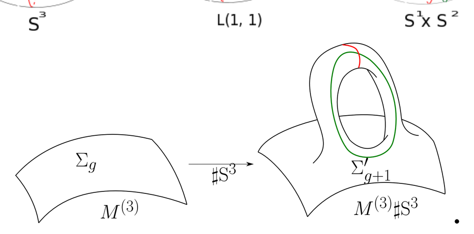


Definition. A **Heegaard splitting** of a compact connected oriented 3-manifold $M^{(3)}$ is the triple (Σ, H_1, H_2), where $\Sigma = \partial H_1 = \partial H_2 = H_1 \cap H_2$, called a Heegard surface, is a compact connected closed oriented 2-dimensional surface, while H_1 and H_2 are 1-handlebodies whose union is $M^{(3)} = H_1 \cup H_2$. $g(\Sigma) = g(H_1) = g(H_2)$. **min**$(g)$ **is a topological invariant**.

One can represent $M^{(3)}$ with a **Heegaard diagram** which consists of a Heegaard surface and α and β attaching curves,e.g.,



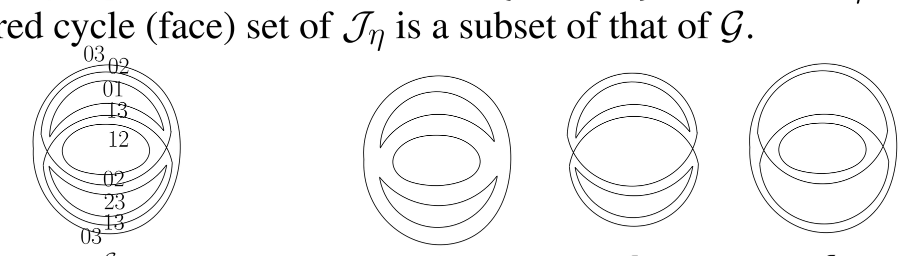
Two diagrams for a given $M^{(3)}$ can be transformed into each other by a finite sequence of moves of stabilisations/handle-cancelations and handle slides. In particular, **stabilisations** can be visualised as



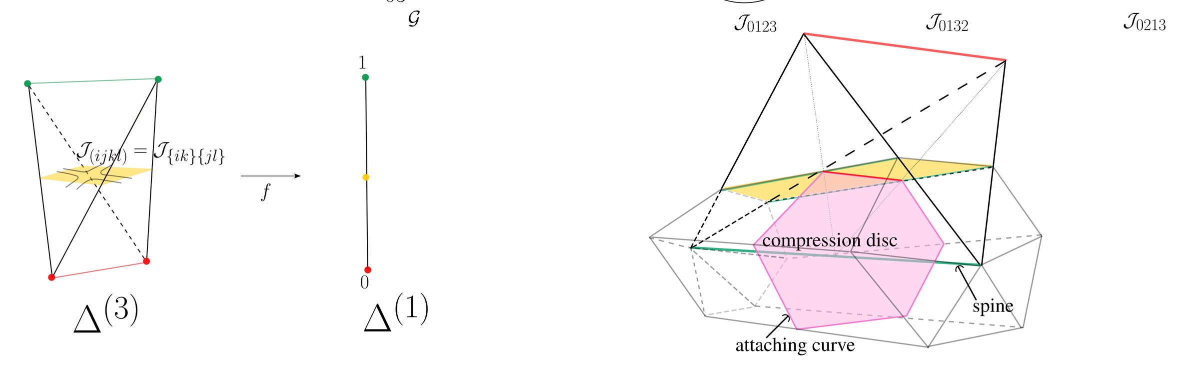
Jackets as Heegaard surfaces (in PL category). [Ryan 2011][Gagliardi 1981]

Given a colored graph $\mathcal{G}$ and the set of its its jackets, we define a combinatorial invariant, called *Gurau degree* (non-negative integer), as the sum of genera of all jackets of $\mathcal{G}$, $\omega(\mathcal{G}) = \sum_{\mathcal{J}} g_{\mathcal{J}}$, where a jacket $\mathcal{J}_\eta$ is an embedded 2-subcomplex of a colored graph $\mathcal{G}$, labeled by a permutation η of the set $\{0, \ldots, d\}$ such that $\mathcal{J}_\eta$ and $\mathcal{G}$ have the same node and line sets and the bicolored cycle (face) set of $\mathcal{J}_\eta$ is a subset of that of $\mathcal{G}$.

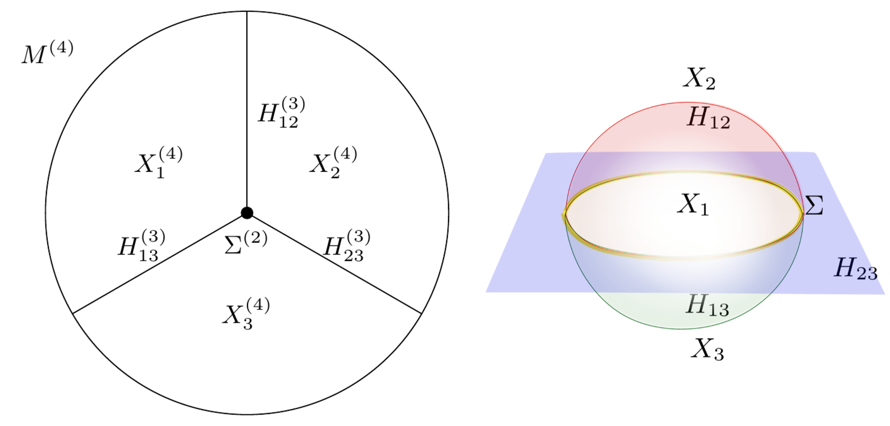
The elementary melon graph for tensor models with 3-indices, and its three jackets, in stranded representation:



Splitting of a 3-simplex (left) and a visualisation of how the 3-dimensional 1-handlebody can be formed (right):



Trisections (in 4 dimensions) in TOP category. [Gay, Kirby 2012]



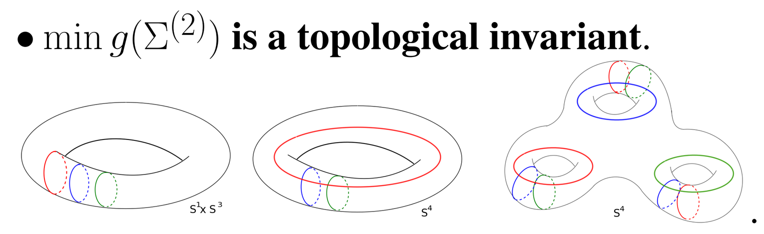
A **trisection** is defined by

- $M^{(4)} = X_1^{(4)} \cup X_2^{(4)} \cup X_3^{(4)}$
- $H_{ij}^{(3)} = X_i^{(4)} \cap X_j^{(4)}$ and $\partial X_i^{(4)} = H_{ij}^{(3)} \cup H_{ik}^{(3)}$.
- $\Sigma^{(2)} = X_1^{(4)} \cap X_2^{(4)} \cap X_3^{(4)}$ is a compact surface.

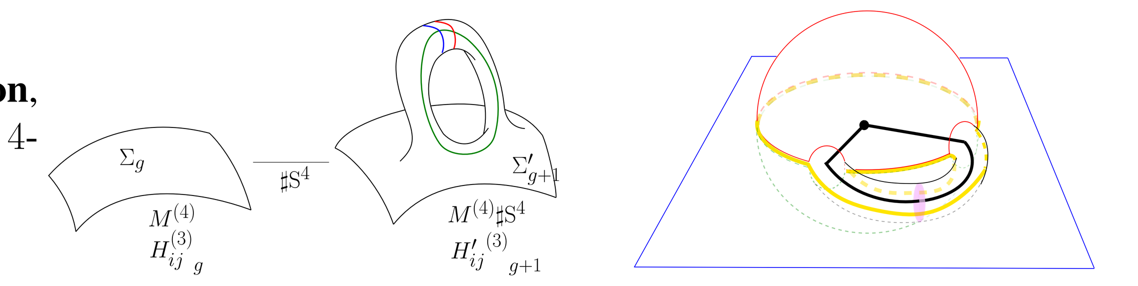
Some remarks follow:

- All $X_i^{(4)}$ and $H_{ij}^{(3)}$ are 1-handlebodies.
- $(H_{ij}^{(3)}, H_{jk}^{(3)}, \Sigma^{(2)})$ forms a Heegaard splitting.
- $g(H_{12}^{(3)}) = g(H_{13}^{(3)}) = g(H_{23}^{(3)}) = g(\Sigma^{(2)})$.
- $\min g(\Sigma^{(2)})$ **is a topological invariant**.

One can represent $M^{(4)}$ in a **trisection diagram** with a central 2-d surface and α, β, γ-attaching curves, e.g.,



Again, one can perform **stabilisation**, which will not alter the overall 4-manifold nor spoil the trisection.

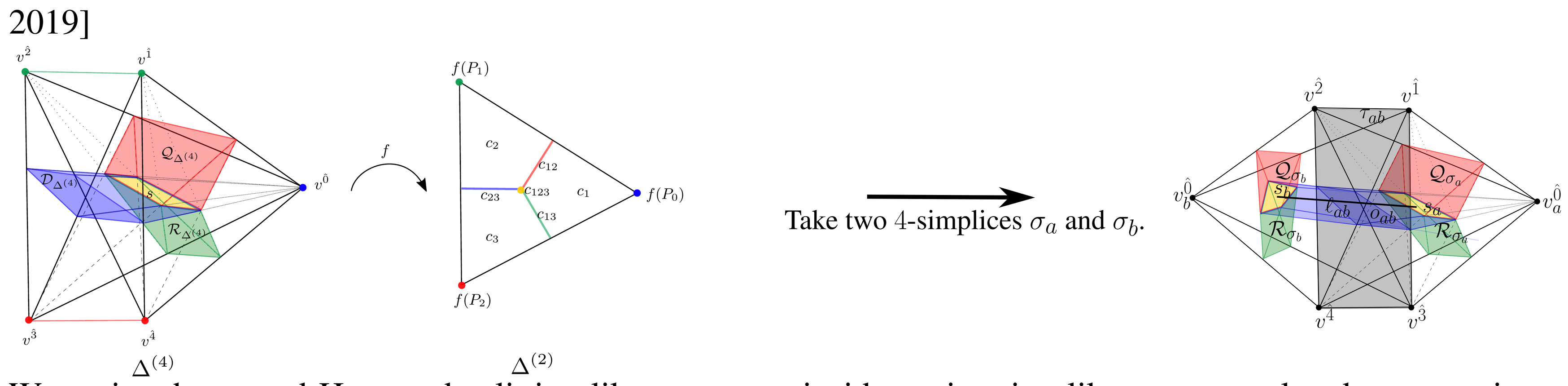


Extending theorem (Montesinos). Given a 4-dimensional 1-handlebody X of genus g and a homeomorphism $\phi : \partial X \to \partial X$, there exists a unique homeomorphism $\Phi : X \to X$ extending ϕ.

Trisections, then, allow us to fully determine $M^{(4)}$ by the three 3-dimensional 1-handlebodies $H_{ij}^{(3)}$ which in turn, can be represented by means of Heegaard diagrams.

Constructing trisections based on colored tensor model graphs in PL category.

1. Start with mapping a 4-simplex $\Delta^{(4)}$ onto a triangle by partitioning the sets of vertices in three sets $P_0 = \{v^{\hat{0}}\}$, $P_1 = \{v^{\hat{1}}, v^{\hat{2}}\}$ and $P_2 = \{v^{\hat{3}}, v^{\hat{4}}\}$ [Bell, Hass, Rubinstein, Tillmann, 2017; Casali, Cristofori, 2019]

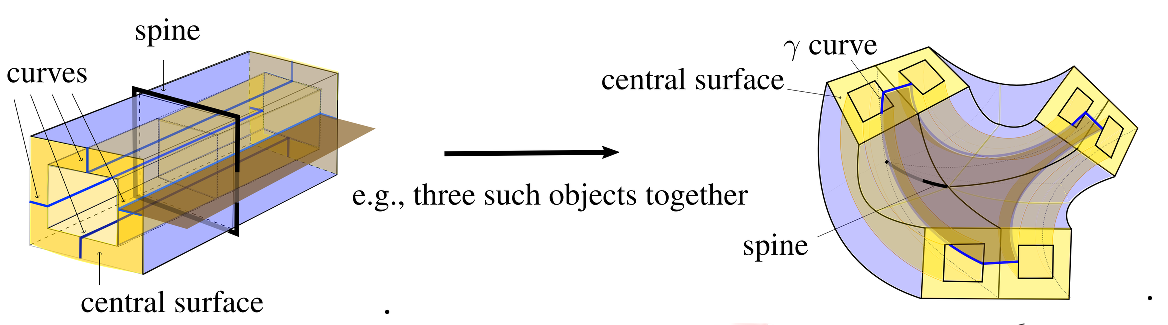


We notice the nested Heegaard splitting-like structures inside a trisection-like structure already present in a single 4-simplex, by pointing out the triples $(s, \mathcal{Q}_{\Delta^{(4)}}, \mathcal{R}_{\Delta^{(4)}})$, $(s, \mathcal{Q}_{\Delta^{(4)}}, \mathcal{D}_{\Delta^{(4)}})$, and $(s, \mathcal{R}_{\Delta^{(4)}}, \mathcal{D}_{\Delta^{(4)}})$. Notice that $s = \Delta^{(4)} \cap K(\mathcal{J}(\mathcal{B}^{\hat{0}}))$, where $K(\mathcal{J}(\mathcal{B}^{\hat{0}}))$ is the realisation of the **jacket of** $\hat{0}$**-bubbles**.

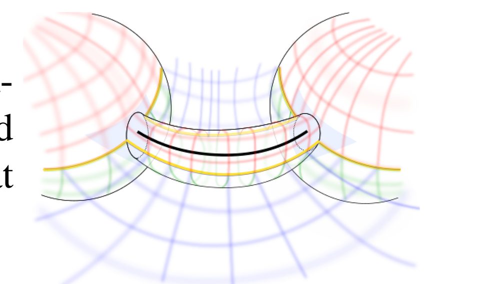
The problem is that each of the 3-dimensional pieces which are to be $H_{ij}^{(3)}$s in a trisection is not connected once we look at the induced structure on a whole colored tensor model graph.

2. So, carve out a neighborhood of 0-color lines of the colored graph which we embed. [Martini, Toriumi 2021].

For example, let us take $\mathcal{D}_{\sigma_a}$ and $\mathcal{D}_{\sigma_b}$, from where we carved out such a neighborhood.



Therefore, we connected the isolated duals of 4-bubbles via the carving operation described above, which really amount to performing boundary connected sum which does not spoil the trisection structure, or stabilisations in the case that the 4-bubbles are connected.

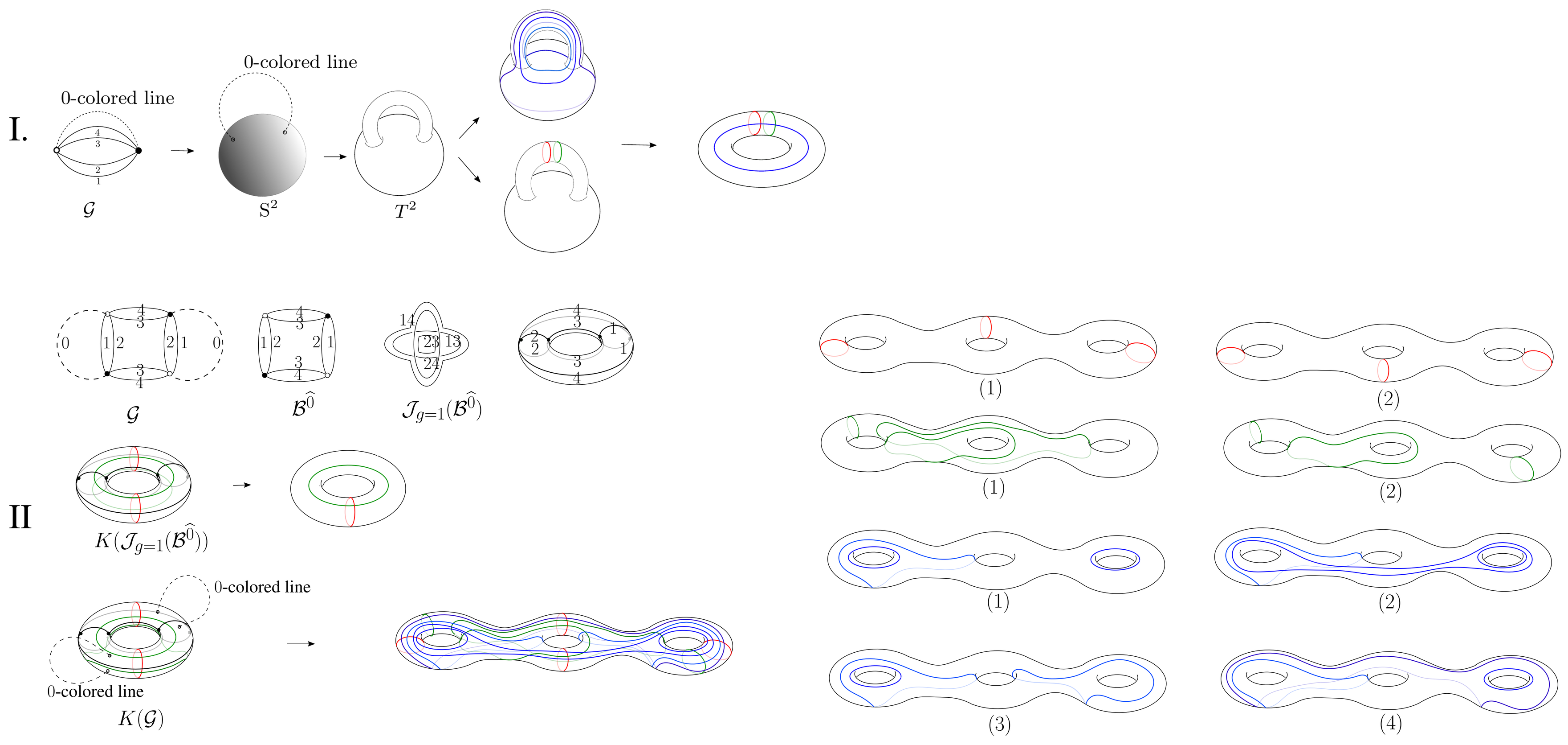


Now let us analyse the genus of the central surface of the trisection just constructed above. Obtain a graph $\tilde{\mathcal{G}}$ derived from a colored graph $\mathcal{G}$, by contracting all the $\hat{0}-$bubbles to points which will then become the nodes of $\tilde{\mathcal{G}}$, whose number is $|\mathcal{V}^{\hat{0}}|$. The **genus of the central surface** is given by

$$g_c = \sum_{a=1}^{|\mathcal{V}^{\hat{0}}|} g_{\mathcal{J}(\mathcal{B}^{\hat{0}}_a)} + L_0 \quad \Rightarrow \quad \sum_{c=1}^{15} g_c = \omega(\mathcal{G}) + 3(4p+1)\,,$$

where L_0 the number of independent loops of $\tilde{\mathcal{G}}$, and $2p$ is the number of nodes of the original colored graph $\mathcal{G}$.

Drawing trisection diagrams based on colored tensor model graphs in PL category.



Conclusions

We formulated a construction on trisections on manifolds realised by colored tensor models with 4-indices. We found in this construction, a central surface of a trisection is realised by a jacket of a $4-$bubble. Some drawbacks in relation to initial hope is that in tensor models as a random geometric approach to quantum gravity, one is interested in taking a continuum limit, where we shall send the number of triangulations to infinity, i.e., the graph will become large, therefore, g_c tends large as well, being far away from the topologically invariant trisection genus. It is also ambiguous whether classifying graphs according to g_c is meaningful.

[1] Pseudo-manifolds here are characterised by non-branching, strongly-connected, and pure to ensure a rather nice property for d-dimensional simplicial complex. However, $K(\mathcal{B}^{\hat{i}})$ may not represent a manifold, i.e., $K(\mathcal{B}^{\hat{i}})$ may not be a sphere.