Reviewing random multimatrix techniques in noncommutative geometry

Random Tensors at CIRM 2022

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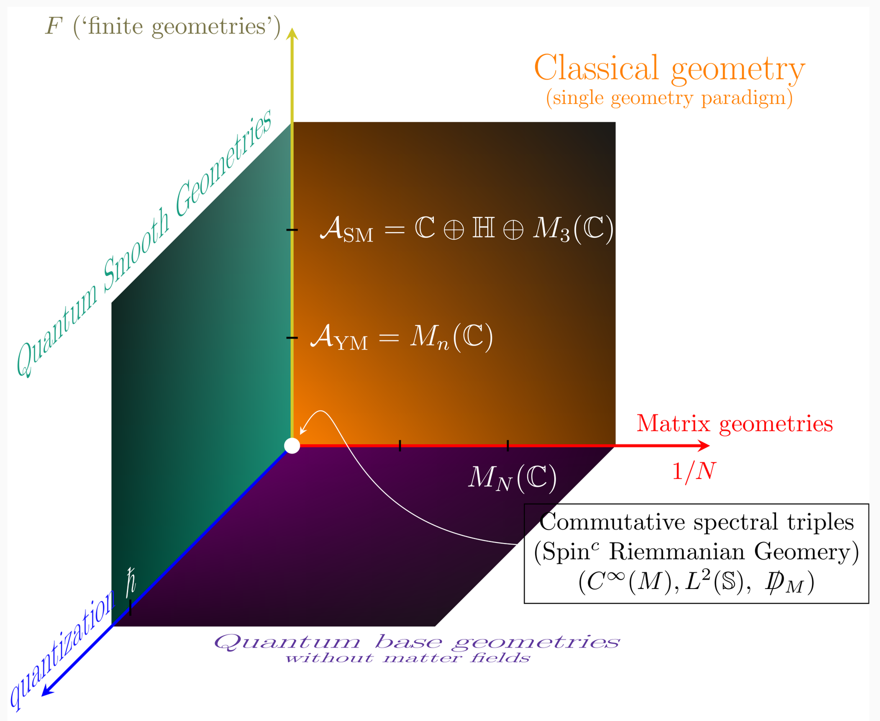
A «Matrix Geometry» Landscape

AIM: quantize NCG $\quad \mathcal{Z}_{\rm NCG} \overset{?}{=} \int_{\rm Dirac} e^{-\frac{1}{\hbar}\operatorname{Tr} f(D)} \mathrm{d}D$



- In noncommutative geometry (or NCG), *spectral triples* $(\mathcal{A},\mathcal{H},D)$—a $*$-algebra $\mathcal{A}$ of bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint operator D—are an abstraction of spin manifolds that allows a non-commutative (nc) $\mathcal{A}$
- $\mathcal{Z}_{\rm NCG}$ well-definable for finite rank D. We use *fuzzy or matrix geometries*, as [Barrett-Glaser *J Phys A* '16]; f polynomial
- Steps: I. Compute the spectral action for fuzzy geometries; II. Define *matrix gauge spectral triples* to add Yang-Mills interactions; III. Renormalization (Continuum limit?)

II. Matrix Yang-Mills Theory

arXiv:2105.01025 (in press)

- spectral action on an almost-commutative (AC) manifold $= M(\text{spin geom.})\times F$ (finite geom.) yields Yang-Mills. The gauge fields are obtained by Morita self-fluctuations
- a *gauge matrix geometry* = matrix spectral triple $\times$ finite spectral triple; the most general (fluctuated) Dirac operator is ($A_\mu \in \Omega^1_D(M_N(\mathbb{C})), c \in M_n(\mathbb{C})_{\rm s.a}$)

$$D = \textstyle\sum_\mu \gamma^\mu \otimes (\overbrace{[L_\mu \otimes 1_n, \cdot\,]}^{l_\mu} + \overbrace{[A_\mu \otimes c, \cdot\,]}^{a_\mu}) + \gamma\otimes\Phi + \overbrace{\sum_{\mu,\nu,\sigma}\gamma^\mu\gamma^\nu\gamma^\sigma \otimes x_{\mu\nu\sigma}}^{\text{(if flat; room for gravitation)}}$$

- the operators l_μ, a_μ serve to define the fuzzy field strength $\mathscr{F}_{\mu\nu} = [l_\mu + a_\mu, l_\nu + a_\nu]$. Here $d_\mu = l_\mu + a_\mu$ is seen as fuzzy analogue of smooth covariant derivative $D_\mu = \partial_\mu + \mathbb{A}_\mu$ ($\mathbb{A}_\mu$, locally, the connection on SU(n)-princ. bundle)
- matrix gauge spectral triples add Yang-Mills fields in the sense that

THEOREM. *The following gauge matrix geometry*

«flat four-dimensional Riemannian fuzzy geometry» $\times\, (M_n(\mathbb{C}), M_n(\mathbb{C}), D_F)$

has the following spectral action, if $f(x) = \sum_m \frac{a_m}{2}x^m$:

$$\frac{1}{4}\operatorname{Tr}_{\mathcal{H}} f(D) = S^\ell_{\rm YM} + S^\ell_{\rm H} + S^\ell_{\text{g-H}} + S^\ell_{2,4} + \text{degree} \geq 5 \text{ operators}$$

Here $S_{2,4}$ are propagators and quartic terms, otherwise each sector is defined as follows:

$$S^\ell_{\rm YM} := -\frac{a_4}{4}\operatorname{Tr}_{M^{\mathbb{C}}_{N\otimes n}}(\mathscr{F}_{\mu\nu}\mathscr{F}^{\mu\nu}), \quad S^\ell_{\rm H} := \operatorname{Tr}_{M^{\mathbb{C}}_{N\otimes n}} f_e(\Phi), \quad S^\ell_{\text{g-H}} := -a_4 \operatorname{Tr}_{M^{\mathbb{C}}_{N\otimes n}}(d_\mu\Phi d^\mu\Phi)\,.$$

- term by term, they are the fuzzy version of $S_{\rm YM}(\mathbb{A}) = -\frac{1}{4}\int_M \operatorname{Tr}_{\mathfrak{su}(n)}(\mathbb{F}_{\mu\nu}\mathbb{F}^{\mu\nu})\mathrm{vol}$, of the Higgs potential, and of the gauge-Higgs coupling $S_{\text{g-H}} = -\int_M D_\mu H (D^\mu H)\mathrm{vol}$
- Gauge symmetry $\mathcal{G} = \mathrm{PU}(N)\times\mathrm{PU}(n)$ is the fuzzy version of the C^∞-gauge group $\mathrm{Diff}(M)\ltimes\mathrm{Maps}(M,\mathrm{SU}(n))$, and gauge invariance due to $\mathscr{F}_{\mu\nu}\longmapsto\mathscr{F}^u = u\mathscr{F}_{\mu\nu}u^*$, $u\in\mathcal{G}$

I. Spectral Action for a Matrix Geometry

arXiv:1912.13288

- *Matrix geometries* of signature (p,q) [Barrett, *J. Math Phys.* '15] are spectral triples with $\mathcal{A} = M_N(\mathbb{C}), \mathcal{H} =$ irreducible $\mathbb{C}\ell(p,q)$-module $V\otimes M_N(\mathbb{C})$.

$$\text{Several axioms imply } D = \sum_a \gamma^a\otimes\{X_a,\,\cdot\,\}_{\epsilon_a} + \sum_a \gamma^a\gamma^b\gamma^c\otimes\{X_{abc},\,\cdot\,\}_{\epsilon_{abc}} + \ldots \qquad \{A,B\}_\pm = AB\pm BA$$

- chord diagrams organize the traces of γ's, e.g. $\operatorname{Tr}_V(\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\rho) = \dim V\cdot(\ldots)$. For a polynomial f, the *spectral action* $\operatorname{Tr} f(D)$ has the form $N\operatorname{Tr}_N P + \operatorname{Tr}_N^{\otimes 2}(Q_{(1)}\otimes Q_{(2)})$ $P, Q_1, Q_2 \in \mathbb{C}_{\langle k\rangle} = \mathbb{C}\langle X_1,\ldots,X_k\rangle$ $(k = 2^{p+q-1})$ where, e.g. for 2d fuzzy geometries (with particular coeffs. depending on p,q and f)

$$P = A^2, B^2, \cancel{AB}, ABAB, AABB, AAABAB, \cancel{ABABAB}, \ldots$$
$$Q_{(1)}\otimes Q_{(2)} = \text{insertions of } \otimes \text{ in such } P\text{'s} = A\otimes A, A\otimes BAB\ldots$$

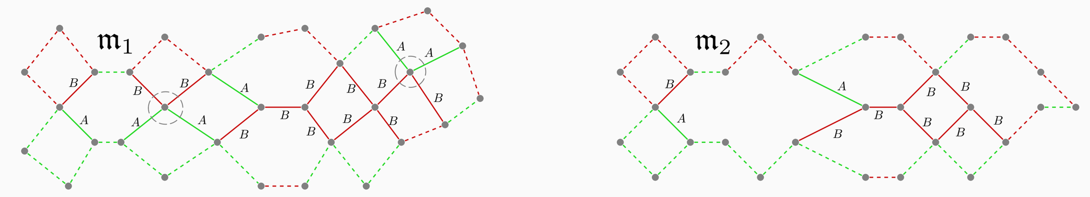


Fig. 1 (Non-stuffed) planar «worded» maps $\mathfrak{m}_1$ and $\mathfrak{m}_2$; Feyman graphs of dim-2 matrix geometries

III. Functional Renormalization: Multimatrix Models (multitraces)

Ann. Henri Poincaré **22** (2021), 3095–3148 (arXiv:2007.10914) as well as arXiv:2111.02858

Physics bit

Quantum theories «flow with energy $t = \log N$». The *effective action* $\Gamma_N[\mathbb{X}]$ describes the theory at scale N, microscopic information on scales $> N$ is washed away. Also, Γ generates 2-edge-connected or 1PI graphs [folklore]

Language

- Let $\mathbb{X} = (X_1,\ldots,X_d)\in M_N(\mathbb{C})^d_{\rm s.a}$ and $\mathbb{C}^{(N)}_{\langle d\rangle} = \mathbb{C}\langle\mathbb{X}\rangle$ or «words»
- [Rota-Sagan-Stein+Voiculescu] nc-derivative $\partial_X : \mathbb{C}_{\langle d\rangle}\to\mathbb{C}^{\otimes 2}_{\langle d\rangle}$ sums over replacements of X in a word by $\otimes$, except at the ends of the word, where one adds 1:

$$\partial_A(PAAR) = P\otimes AR + PA\otimes R\,,$$
$$\text{but}\quad \partial_A(ALGEBRA) = 1\otimes LGEBRA + ALGEBR\otimes 1\,.$$

Also ∂_A on traces yields the *cyclic derivative*: $\partial_A\operatorname{Tr}(PAAR) = ARP + RPA$, for instance. The *nc-Hessian* is the matrix with entries $\mathrm{Hess}_{a,b}\operatorname{Tr}P = \partial_{X_a}\partial_{X_b}\operatorname{Tr}P$.

EXAMPLE. $\mathrm{Hess}\{\operatorname{Tr}(ABAB)\}$ reads then

$$\begin{pmatrix}\partial^A\circ\partial^A & \partial^A\circ\partial^B\\ \partial^B\circ\partial^A & \partial^B\circ\partial^B\end{pmatrix}\operatorname{Tr}(ABAB) = 2\begin{pmatrix} & B\otimes B & AB\otimes 1 + 1\otimes BA\\ & BA\otimes 1 + 1\otimes AB & A\otimes A\end{pmatrix}$$

- the presence of multitraces (see Fig. 2) extends this algebra to $\mathcal{B} = \mathcal{A}^{(N)}_d = \mathbb{C}^{\otimes 2}_{\langle d\rangle}\oplus\mathbb{C}^{\boxtimes 2}_{\langle d\rangle}$ with the product $\star$ given by

$$(U\otimes W)\star(P\otimes Q) = PU\otimes WQ\,,\qquad (U\boxtimes W)\star(P\otimes Q) = U\boxtimes PWQ\,,$$
$$(U\otimes W)\star(P\boxtimes Q) = WPU\boxtimes Q\,,\qquad (U\boxtimes W)\star(P\boxtimes Q) = \operatorname{Tr}(WP)U\boxtimes Q$$

for $P,Q,U,W\in\mathbb{C}_{\langle d\rangle}$. Traces: $\operatorname{Tr}_{\mathcal{B}}(P\otimes Q) = \operatorname{Tr}P\operatorname{Tr}Q$, $\operatorname{Tr}_{\mathcal{B}}(P\boxtimes Q) = \operatorname{Tr}(PQ)$.

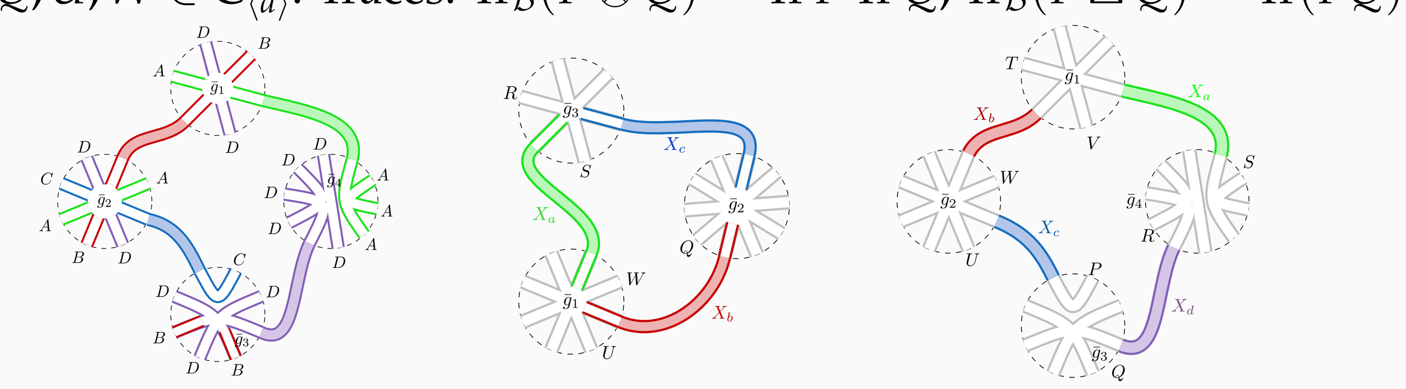


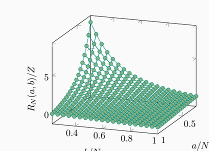
Fig. 2 Examples of graphs. From left to right: a graph of a 4-matrix model whose effective vertex is $\operatorname{Tr}(BDBD^7)\operatorname{Tr}(A^3DACDBACDADB)$. Next two graphs are both 1-loop (in the QFT sense) but only the one in the middle also in the topological sense. The latter is a contribution to $\mathrm{Hess}_{a,b}O_1\star\mathrm{Hess}_{b,c}O_2\star\mathrm{Hess}_{c,d}O_3\star\mathrm{Hess}_{d,a}O_4$

Renormalization Group: how do words flow

- $\Gamma_N = \sum_i \bar{g}_i\operatorname{Tr}P_i + \sum_i \bar{g}_{i,j}\operatorname{Tr}^{\otimes 2}(Q_{1,i}\otimes Q_{2,j}) + \ldots$, cf. I.
- unrenormalized couplings $\bar{g}_i, \bar{g}_{i,j},\ldots$ depend on N, renormalized: $g_i = \alpha_i(N)\bar{g}_i(N), g_{i,j} = \alpha_{i,j}(N)\bar{g}_{i,j}(N),\ldots$

THEOREM.(«FRG for multiMM») *Wetterich eq. holds*

$$\partial_t\Gamma_N[\mathbb{X}] = \frac{1}{2}\operatorname{Tr}_{M_d(\mathcal{B})}\left(\frac{\partial_t R_N}{\mathrm{Hess}\,\Gamma_N[\mathbb{X}] + R_N}\right)$$



- RHS $\in\mathbb{C}[[\{g_i,g_{i,j},\ldots\}_{i,j,\ldots}]][[\{O_i,O_{i,j},\ldots\}_{i,j,\ldots}]]$, only a formal series for the time being, is understood as a geometric series in $\mathrm{Hess}\,\Gamma_N\in M_d(\mathcal{A}^{(N)}_d,\star)$
- LHS determines the β-functions $\beta_w = \partial_t[g_w(N)]$, which are determined from $[O_w]$ RHS

EXAMPLE. Modulo $\eta = \partial_t Z$-coeffs, up to double-traces and cubic terms:

$$\beta(g_{ABBA}) - g_{ABBA}(2\eta+1) \sim g_{AAAA}\times g_{ABBA} + g_{BBBB}\times g_{ABBA} + (g_{ABAB})^2 + (g_{ABBA})^2$$

- $(2,0)$-*geometry*: β-functions for 48 operators are found and numerically solved: among ~ 600 real-valued solutions, the unique one with a single relevant direction yields $g^{\rm crit.}_{AAAA} = 1.002\cdot g^{\text{Kazakov Zinn-Justin}}_{AAAA}\sim 1/4\pi$

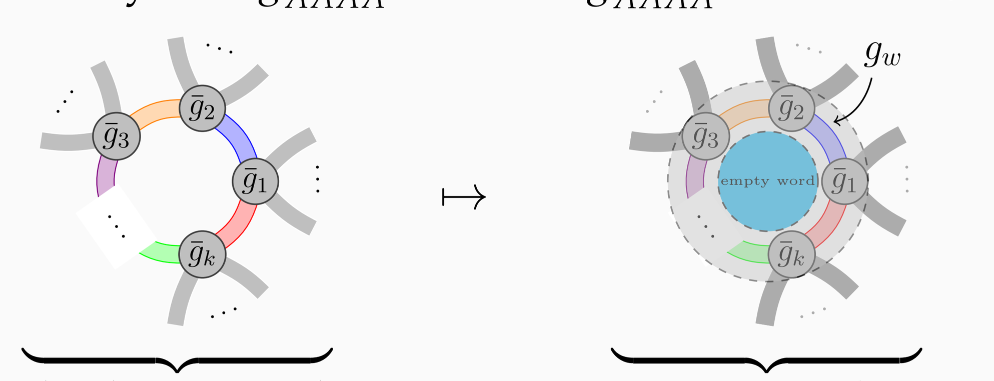


Fig. 3 How the one-loop structure of the FRG is encoded in $M_d(\mathcal{A}_d,\star)$. *Left:* Unrenormalized interactions $\bar{g}_i$ appearing in a k-th power of the Hessian. *Right:* The contribution to the β_w-function, w formed by reading off clockwise the legs.

To download poster: https://www.thphys.uni-heidelberg.de/~perez/carlos.html