Abstract Operator Systems:

Where Quantum Information Theory meets Free Semialgebraic Geometry

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Motivation:

- Abstract operator systems (AOS) originally from operator algebra
- similar concept appears in free semialgebraic geometry and free convexity
- many concepts from quantum information can be studied in this framework

$\rightsquigarrow$ interesting geometric objects + (possible) applications

Definition:

V $\mathbb{C}$-vectorspace with involution

$M_s(\mathbb{C}) \otimes V = M_s(V)$ inherits involution

$$C_s \subseteq \left(M_s(\mathbb{C}) \otimes V \right)_{her} = Her_s(V)$$

proper convex cone, for each $s \geq 1$, s.t.

$$\forall X \in C_s, \forall W \in Mat_{s,t}(\mathbb{C}): \quad V^* X V \in C_t$$

$$\boxed{C = (C_s)_{s \geq 1} \text{ abstract operator system}}$$

Example: $V = \mathrm{Mat}_d(\mathbb{C})$

$$\mathrm{Mat}_s(\mathbb{C}) \otimes \mathrm{Mat}_d(\mathbb{C}) = \mathrm{Mat}_s(\mathrm{Mat}_d(\mathbb{C})) = \mathrm{Mat}_{s\cdot d}(\mathbb{C})$$

so $\quad C_s \subseteq \mathrm{Her}_s(\mathbb{C}) \otimes \mathrm{Her}_d(\mathbb{C}) = \mathrm{Her}_{s\cdot d}(\mathbb{C})$

e.g.:

$C_s = \mathrm{Pos}_s := \left\{ X = \sum_i A_i \otimes B_i \;\middle|\; X \geq 0 \right\}$ positive matrices/states/cp maps

Important feature: d is fixed, but we consider all $s \geq 1$ simultaneously

more examples from quantum information:

$$PPT_s := \left\{ \sum_i A_i \otimes B_i \;\middle|\; \sum_i A_i \otimes B_i^T \geq 0 \right\}$$

matrices with positive partial transpose

$$Sep_s := \left\{ \sum_i A_i \otimes B_i \;\middle|\; A_i \geq 0,\ B_i \geq 0 \right\}$$

separable states / entanglement breaking maps

$$BPos_s := \left\{ X = \sum_i A_i \otimes B_i \;\middle|\; X \text{ block positive} \right\}$$

block positive matrices / positive maps

Smallest / largest AOS:

$C \subseteq V_{\mathrm{her}}$ proper convex cone

$$C_s^{\min} := \Big\{ \sum_i P_i \otimes c_i \;\Big|\; c_i \in C,\ P_i \in \mathrm{Her}_s(\mathbb{C}),\ P_i \geq 0 \Big\}$$

$C^{\min} = \left(C_s^{\min} \right)_{s \geq 1}$ smallest AOS with $C_1 = C$.

$$C_s^{\max} := \Big\{ X \in \mathrm{Her}_s(V) \;\Big|\; \forall v \in \mathbb{C}^s : v^* X v \in C \Big\}$$

$C^{\max} = \left(C_s^{\max} \right)_{s \geq 1}$ largest AOS with $C_1 = C$

Example:

$$C = \mathrm{cone}\{v_1, \dots, v_n\} = \{a \mid \ell_1(a) \geq 0, \dots, \ell_m(a) \geq 0\}$$

finitely generated/polyhedral cone

$$C_S^{min} = \left\{ \sum_{i=1}^{n} P_i \otimes v_i \;\middle|\; P_i \geq 0 \right\}$$ polytope extension

$$\cap\!\mathrm{I}$$

$$C_S^{max} = \{ A \in \mathrm{Her}_S(V) \mid \ell_1(A) \geq 0, \dots, \ell_m(A) \geq 0 \}$$ polyhedral extension

Example:

$$V = \mathrm{Mat}_d(\mathbb{C}), \quad C = \mathrm{Psd}_d \subseteq \mathrm{Her}_d(\mathbb{C})$$

Then $\quad C^{\min} = \mathrm{Sep}$

$$C^{\max} = \mathrm{BPos}.$$

Definition: Concrete Operator System

H Hilbert space

$V \subseteq B(H)$ unital $*$-subspace COS

$\ulcorner$ $\mathrm{Mat}_s(V) \subseteq \mathrm{Mat}_s(B(H)) = B(H^s)$

$$C_s := B(H^s)_{psd} \cap \mathrm{Her}_s(V)$$

this structure comes for free !! $\lrcorner$

Theorem [Ruan '88] AOS = COS

If $(C_s)_{s\geq 1}$ is an AOS on V, then

$$\exists\ \varphi : V \longrightarrow B(H) \quad *\text{-linear}$$

s.t. $$C_s = (\mathrm{id}_s \otimes \varphi)^{-1}\big(B(H^s)_{psd}\big).$$

Definition:

An AOS has a finite dimensional realisation

if the Hilbert space H from Ruan's Theorem

can be chosen finite dimensional.

This is dual to the AOS being

finitely generated.

$\boxed{\text{noncommutative polyhedron}}$

Example:

Pos and PPT have a f.d. realisation and are finitely gen.

For Sep and BPos neither holds.

Free Spectrahedron = AOS with f.d. realization

$$V = \mathbb{C}^n, \quad \mathrm{Her}_s(\mathbb{C}^n) = \mathrm{Her}_s(\mathbb{C})^n$$

$$\ell : \mathbb{C}^n \longrightarrow \mathrm{Mat}_m(\mathbb{C})$$

$$(\mathrm{id}_s \otimes \ell)(A_1, \dots, A_n) = \ell(e_1) \otimes A_1 + \dots + \ell(e_n) \otimes A_n$$

$$= M_1 \otimes A_1 + \dots + M_n \otimes A_n \geq 0$$

linear matrix inequality

defines a free spectrahedron.

Theorem [Fritz, N. & Thom; Passer, Shalit & Solel; Huber & N.]

(i) $C^{\min} = C^{\max} \iff C$ simplex cone

(ii) C polyhedral but not simplex, then

$C^{\min}$ finitely generated but no f.d. realization

$C^{\max}$ not f.g. but with a f.d. realization

In particular $C^{\min} \subsetneq C^{\max}$, already for $s \geq 2$.

Corollary [Cariello; De las Cuevas, Drescher & N.]

Every bipartite state of rank 2 is separable.

MPDO bond dim 2

Theorem [Berger, Drescher & N.]

$\boxed{\text{Wigner type result}}$

(i) If $\mathcal{C} = (C_s)_{s \geq 1}$ is an AOS with $C_1 = \mathrm{Psd}_d(\mathbb{C})$ which has a f.d. realization, then

$$\mathcal{C} \subseteq \mathrm{Pos} \quad \text{or} \quad \mathcal{C} \subseteq \mathrm{PPT}.$$

(ii) If $\mathcal{C}$ is a f.g. AOS with $C_1 = \mathrm{Psd}_d(\mathbb{C})$ then

$$\mathrm{Pos} \subseteq \mathcal{C} \quad \text{or} \quad \mathrm{PPT} \subseteq \mathcal{C}.$$

(iii) The only AOS with $C_1 = \mathrm{Psd}_d(\mathbb{C})$ which have a f.d. realization and are f.g. are Pos and PPT.

Choi type results fail

<u>Corollary</u> [Berger & N.]

$\mathrm{PPT} \cap \mathrm{Pos}$ is not finitely generated.

$\mathrm{Decomp} := \mathrm{Pos} + \mathrm{PPT}$ has no f.d. realization.

just interesting

<u>Theorem</u> [Berger, Drescher & N.]

Over $C_1 =$  $\subseteq \mathbb{R}^3$ there is a chain of AOS with f.d. realization, with limit C^{max}, which has no f.d. realization.

Theorem [Blum & Nechita]

$B_1, \ldots, B_d \in \mathrm{Her}_m(\mathbb{C})$ are jointly measurable

if and only if

a certain AOS with f.d. realisation

(defined by $B_1, \ldots, B_d$) contains the

largest operator system over the

matrix diamond.

Quantum Magic Squares:

$\boxed{\text{fun!}}$

$A = (A_{ij})_{i,j=1,\dots,n} \in \mathrm{Mat}_n(\mathrm{Psd}_s(\mathbb{C}))$ quantum magic square

if $\sum_i A_{ij} = I_s$ and $\sum_j A_{ij} = I_s$

for all $i, j = 1, \dots, n$.

Its called a quantum permutation matrix

if $A_{ij}^2 = A_{ij}$ in addition.

Remarks:

(i) A is a quantum magic square iff every column and row contains a POVM.

(ii) A is a quantum permutation matrix iff every column and row contains a projective measurement.

(iii) $A = (A_{ij})_{i,j=1,\dots,n}$ quantum permutation matrix
V isometry $\rightsquigarrow (V^* A_{ij} V)_{ij}$ quantum magic square

Question [Quantum Birkhoff - von Neumann, Magic Naimark]

Is every quantum magic square of the form

$$\left(V^* A_{ij} V\right)_{ij}$$

for some quantum permutation matrix

$$\left(A_{ij}\right)_{ij} \quad ?$$

„inner size" n is **not** fixed!

AOS can help:

V = space of $n \times n$ complex matrices with constant row and column sums

$*$ = entrywise conjugation

C_1 = cone of matrices with nonnegative entries $\subseteq V_{her}$

$\mathrm{Her}_s(V) = \mathrm{Mat}_n(\mathrm{Her}_s(\mathbb{C}))$

C^{max} ~ quantum magic squares

Theorem [De las Cuevas, Drescher & N.]

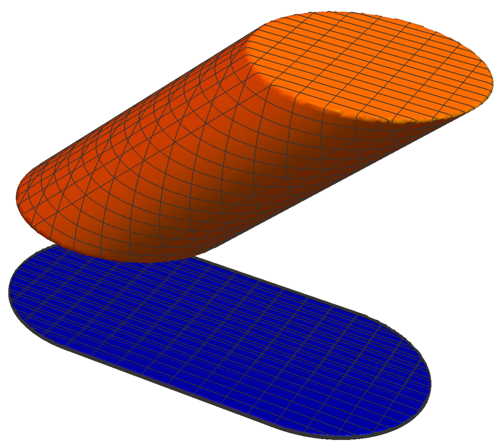
The AOS generated by quantum permutation matrices is strictly contained in $C^{\max}$.

There exist quantum magic squares which do not dilate to a quantum permutation matrix.

⌜For $n=3$ this follows since quantum permutations generate $C^{\min}$ and C_1 is not simplicial⌟

Advertisement !!!

Geometry of Linear Matrix Inequalities



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available soon...

Thank you for your

attention !!