Annihilating Entanglement

&

Tensor Radii of Banach Spaces

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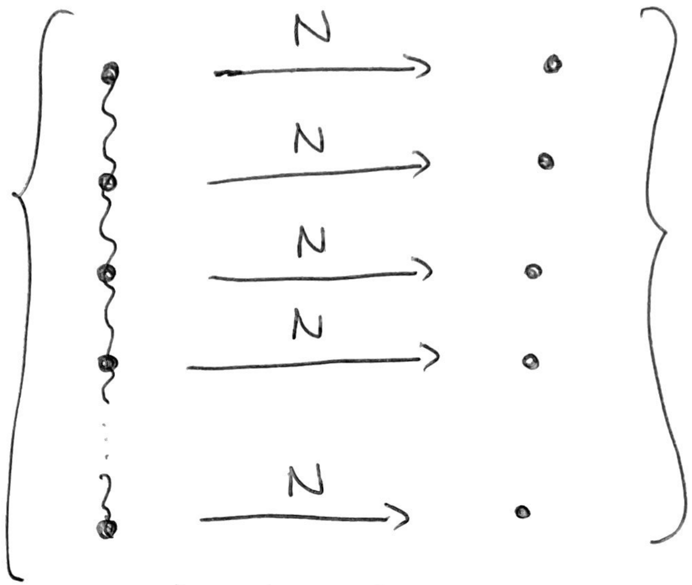
Motivation

Which quantum channels annihilate entanglement?

What is entanglement annihilation?

Entangled q-state of many-body system



Fully separable q-state

Local action of noise N

Def: A q-channel $N : M_{d_A} \to M_{d_B}$ is called <u>entanglement annihilating</u> (EA) if

$$N^{\otimes k}(\rho) \subseteq \mathrm{conv}\left\{ \sigma_1 \otimes \sigma_2 \otimes \dots \otimes \sigma_k \ : \ \sigma_i \geq 0, \mathrm{Tr}(\sigma_i) = 1 \right\}$$

$$\forall \rho \in PSD\left((\mathbb{C}^{d_A})^{\otimes k} \right) \quad \forall k \in \mathbb{N}.$$
$$(\mathrm{Tr}(\rho) = 1)$$

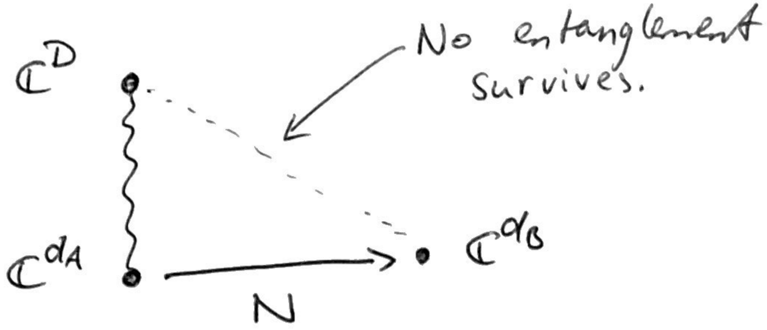
Known examples of EA channels: Entanglement breaking channels

Def: A q-channel $N : M_{d_A} \to M_{d_B}$ is called <u>entanglement breaking</u> (EB) if

$$N(X) = \sum_{i=1}^{\ell} \mathrm{Tr}[A_i X]\, B_i$$

for $A_i, B_i \geq 0$.

<u>Observation:</u> EB $\Longrightarrow$ EA (Proof later)



$$\boxed{\begin{array}{l}\underline{\text{Big open problem:}} \\ \qquad \mathrm{EA} \overset{?}{\Longrightarrow} \mathrm{EB}\end{array}}$$

$\leadsto$ If answer is no, then several open problems in QIT (Distillation problem, PPT^2-conj. TS-positive maps) are solved.

Known: For q-channels $N: M_2 \rightarrow M_{d_B}$

& $N: M_{d_A} \rightarrow M_2$ we have

$$EA = EB.$$

$\rightsquigarrow$ Closely related to solution of distillation problem if either dimension is 2.

What is this talk about?

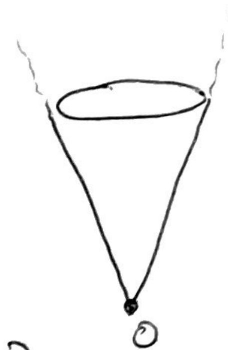
(1) Formulate this problem in the general theory of cones in f.d. vector spaces.

(2) Solve it for a class of cones.

(3) Connections to Banach space theory.

The Setting:

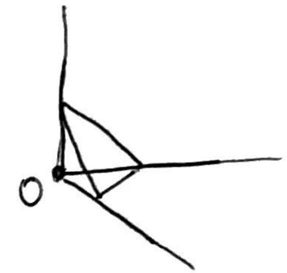
$C \subset V$ proper cone in f.d. (real) vector space V



$\hookrightarrow$ Closed; Generating: $C - C = V$; Pointed: $C \cap (-C) = \{0\}$

Dual cone: $C^* = \{ \varphi \in V^* : \varphi(x) \geq 0 \ \forall x \in C \}$

Note: $(C^*)^* = C$



Examples:

- $\mathbb{R}^n_+ = \{ (x_1, \dots, x_n) : x_1, \dots, x_n \geq 0 \}$ classical cones
- $PSD(\mathbb{C}^d) \subset Her(\mathbb{C}^d)$ quantum cones
- For normed space $X = (\mathbb{R}^d, \|\cdot\|)$

$$C_X = \{ (t, x) \in \mathbb{R} \times X : \|x\| \leq t \} .$$

$$\rightsquigarrow C_X^* = C_{X^*}$$

- Lorentz cones $L_n = C_{\ell_2^n}$

Entanglement between cones

Natural tensor products:

<u>Minimal</u>: $C_1 \otimes_{\min} C_2 = \mathrm{cone}\{x \otimes y : x \in C_1, y \in C_2\}$

<u>Maximal</u>: $C_1 \otimes_{\max} C_2 = \left(C_1^* \otimes_{\min} C_2^*\right)^*$

$\Big[\; z \in C_1 \otimes_{\max} C_2 \iff (\varphi_1 \otimes \varphi_2)(z) \geq 0 \quad \forall \varphi_1 \in C_1^*, \forall \varphi_2 \in C_2^*.$

<u>Definition</u>: Any $z \in C_1 \otimes_{\max} C_2 \setminus C_1 \otimes_{\min} C_2$ is called <u>entangled</u>.

<u>Thm:</u> (Aubrun, Lami, Palazuelos, Plavala)

Entanglement exists iff neither $C_1 \simeq \mathbb{R}^n_+$ nor $C_2 \simeq \mathbb{R}^n_+$.

classical cones

Minimal & maximal tensor products can be iterated

$$C^{\otimes_{\min} k} = \operatorname{cone}\{ x_1 \otimes \cdots \otimes x_k \;:\; x_1, \ldots, x_k \in C \}$$

$\rightsquigarrow$ "Fully separable tensors"

$$C^{\otimes_{\max} k} = \left((C^*)^{\otimes_{\min} k} \right)^* .$$

$\rightsquigarrow$ $z \in C^{\otimes_{\max} k} \iff (\varphi_1 \otimes \cdots \otimes \varphi_k)(z) \geq 0$

$\forall \varphi_1, \ldots, \varphi_k \in C^* .$

Classes of linear maps:

$C_1 \subset V_1$ proper cones
$C_2 \subset V_2$

$P: V_1 \to V_2$ linear is called

- (C_1, C_2)-positive if $P(C_1) \subseteq C_2$
- (C_1, C_2)-entanglement breaking (EB) if $P = \sum_i \psi_i \varphi_i$ with $\psi_i \in C_2$, $\varphi_i \in C_1^*$

  (Equivalently: $(id_V \otimes P)(C \otimes_{max} C_1) \subseteq C \otimes_{min} C_2$ $\forall$ proper cones C)

- (C_1, C_2)-entanglement annihilating (EA) if

$$P^{\otimes k}\left(C_1^{\otimes_{max} k}\right) \subseteq C_2^{\otimes_{min} k}.$$

As before: $EB \Longrightarrow EA$

For $P = \sum_i y_i \varphi_i$ with $y_i \in C_2$, $\varphi_i \in C_1^*$ we have

$$P^{\otimes k}(z) = \underbrace{\sum_{i_1 \ldots i_k} y_{i_1} \otimes \ldots \otimes y_{i_k} \underbrace{(\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_k})(z)}_{\geq 0 \text{ if } z \in C_1^{\otimes_{\max} k}}}_{\in C_2^{\otimes_{\min} k}}.$$

$$\boxed{\text{When is } EB = EA \text{ ?}}$$

Definition: (Resilience)

(1) (C_1, C_2) is called resilient if every (C_1, C_2)-EA map is (C_1, C_2)-EB.

(2) C is called resilient if (C, C) is resilient according to (1).

Basic properties:

- Classical cones are resilient.
- (C_1, C_2) resilient $\Longrightarrow$ (C_2^*, C_1^*) resilient
- C_1 resilient $\Longrightarrow$ (C_1, C_2) and (C_2, C_1) are resilient $\forall\, C_2$.

Known: $PSD(\mathbb{C}^2)$ is resilient.

$\rightsquigarrow$ Difficult to generalize to $PSD(\mathbb{C}^d)$

Observation: $PSD(\mathbb{C}^2) \simeq L_3$

$$\boxed{\text{Can we generalize resilience to Lorentz cones?}}$$

Main result: L_n is resilient $\forall n \in \mathbb{N}$.

Corollary: (C, L_n) and (L_n, C) are resilient $\forall C$.

How to prove that every L_n is resilient?

1. Reduce problem to a "nice" class of maps
2. Apply these maps to certain highly entangled tensors
3. Conclude that maps have to be EB in order to annihilate entanglement.

1 & 2 work more general for cones $C_X = \{(t,x) : t \geq \|x\|\}$ for normed spaces $X = (\mathbb{R}^n, \|\cdot\|)$.

$\rightsquigarrow$ Interesting connection to Banach space theory.

Resilience of $C_X = \{(t,x) : t \geq \|x\|\}$.

Central maps: $\begin{cases} P = \alpha \oplus T , & \alpha \in \mathbb{R}, T : X \to X \text{ linear} \\ P((t,x)) = (\alpha t, T(x)) \end{cases}$

Central map $P = \alpha \oplus T$ is

- (C_X, C_X)-positive if $\|T\|_{X \to X} \leq \alpha$.
- (C_X, C_X)-EB if $\|T\|_{N(X\to X)} \leq \alpha$

<u>Nuclear norm:</u>

$$\|T\|_{N(X\to X)} = \inf \sum_i \|y_i\|_X \|x_i^*\|_{X^*} \quad \text{s.t. } T = \sum_i y_i x_i^* .$$

$$\boxed{\underline{\text{Lemma}}: \; C_X \text{ resilient} \iff \text{Every EA central map is EB}}$$

When are central maps EA ?

Natural tensor norms on $X^{\otimes k}$:

X^*-unit ball

- $\|z\|_{\varepsilon_k(X)} = \sup\left\{ \left|(\lambda_1 \otimes \dots \otimes \lambda_k)(z)\right| : \lambda_1, \dots, \lambda_k \in B_{X^*} \right\}$

<u>injective norm</u>

- $\|z\|_{\pi_k(X)} = \inf\left\{ \sum_i \|x_i^{(1)}\| \dots \|x_i^{(k)}\| : z = \sum_i \underset{X}{\underset{\cap}{x_i^{(1)}}} \otimes \dots \otimes \underset{X}{\underset{\cap}{x_i^{(k)}}} \right\}$

<u>projective norm</u>

<u>Observation</u>:

$$\boxed{\left.\begin{aligned} C_{\varepsilon_k(X)} &= C_X^{\otimes_{\max} k} \cap X_k \\ C_{\pi_k(X)} &= C_X^{\otimes_{\min} k} \cap X_k \end{aligned}\right\} \text{ for } X_k = \operatorname{span}\left(\{e_0^{\otimes k}\} \cup X^{\otimes k}\right)}$$

<u>Note</u>: $C_X = \{(t, x) : t \geq \|x\|\}$

$e_0 = (1, 0)$.

When are central maps EA?

$$C_{\varepsilon_k(X)} = C_X^{\otimes_{\max} k} \cap X_k \qquad X_k = \operatorname{span}\left(\{e_0^{\otimes k}\} \cup X^{\otimes k}\right)$$

$$C_{\pi_k(X)} = C_X^{\otimes_{\min} k} \cap X_k$$

If $P = \alpha \oplus T$ is EA, then

$$\underbrace{P^{\otimes k}\left(C_{\varepsilon_k(X)}\right) \subseteq C_{\pi_k(X)} \quad \forall k \in \mathbb{N}.}$$

$$\iff \quad \|T^{\otimes k}\|_{\varepsilon_k(X) \to \pi_k(X)} \leq \alpha^k .$$

$\boxed{\begin{aligned} &\underline{\text{Def:}}\ (\text{Tensor radius}) \\ &\tau_\infty(T) = \sup_{k \in \mathbb{N}} \|T^{\otimes k}\|^{1/k}_{\varepsilon_k(X) \to \pi_k(X)} \\ &\qquad\ \ = \lim_{k \to \infty} \|T^{\otimes k}\|^{1/k}_{\varepsilon_k(X) \to \pi_k(X)} \end{aligned}}$

$\boxed{\begin{aligned} &\underline{\text{Lemma:}} \\ &\text{If } \alpha \oplus T \text{ is EA, then} \\ &\qquad \tau_\infty(T) \leq \alpha . \end{aligned}}$

Summary so far:

Central map $P = \alpha \oplus T$ is

- EB $\iff$ $\|T\|_{N(X \to X)} \leq \alpha$
- EA $\implies$ $\tau_\infty(T) \leq \alpha$

Tensor radius: $\tau_\infty(T) = \lim_{k \to \infty} \|T^{\otimes k}\|^{1/k}_{\varepsilon_k(X) \to \pi_k(X)}$

Easy to see: $\|T\|_{X \to X} \leq \tau_\infty(T) \leq \|T\|_{N(X \to X)}$

Observation: If $\tau_\infty(T) = \|T\|_{N(X \to X)}$ $\forall T : X \to X$,

then $\underbrace{\text{every central EA map is EB.}}$

$\implies C_X$ resilient.

Definition: $X = (\mathbb{R}^n, \|\cdot\|)$ has the <u>nuclear tensorization property</u> (NTP) if $\tau_\infty(T) = \|T\|_{N(X\to X)}$ $\forall T: X \to X$.

$$\begin{array}{ccc} X \text{ has NTP} & \Longrightarrow & C_X \text{ resilient} \\ \Big\Updownarrow \text{ Theorem*} & & \\ X \text{ is Euclidean} & & \end{array}$$

Corollary: $L_n = C_{\ell_2^n}$ is resilient $\forall n \in \mathbb{N}$.

$\boxed{\begin{array}{l}\underline{\text{Theorem}^*}\text{: The following are equivalent.} \\ (1)\quad \tau_\infty(T) = \|T\|_{N(X\to X)} \quad \forall T: X \to X \\ (2)\quad X \text{ Euclidean}\end{array}}$

<u>Proof of (2) $\Rightarrow$ (1)</u>:

Two steps:

1.) Show for $T = id_X$

$\rightsquigarrow \tau_\infty(id_X) = \dim(X)$

2.) Symmetrization to reduce general T to id_X.

1.) Show $\tau_\infty(id_X) = \dim(X) =: n$

From before: $\tau_\infty(id_X) \leq n$.

Show: $\tau_\infty(id_X) \geq n$

$$\tau_\infty(id_X) = \lim_{k\to\infty} \sup_{z \in X^{\otimes k}} \left(\frac{\|z\|_{\pi_k(X)}}{\|z\|_{\varepsilon_k(X)}} \right)^{1/k}$$

Consider orthogonal $A_1, \dots, A_n \in M_N(\mathbb{R})$

s.th.

$$A_i^T A_j + A_j^T A_i = 0 \quad \text{when } i \neq j$$

$\rightsquigarrow$ exist for N large enough. (Hurwitz-Radon problem)

Define:
$$\boxed{\begin{gathered} z(i,j) = \sum_{l_1 \dots l_k} [A_{l_1} \cdots A_{l_k}]_{ij} \, e_{l_1} \otimes \dots \otimes e_{l_k} \\ \forall i,j \in \{1, \dots, N(n)\}. \end{gathered}}$$

$$z(i,j) = \sum_{\ell_1 \dots \ell_k} [A_{\ell_1} \cdots A_{\ell_k}]_{ij} \; e_{\ell_1} \otimes \dots \otimes e_{\ell_k}$$

Lemma:

(1) $\forall i,j \; : \; \| z(i,j) \|_{\mathcal{E}_k(X)} \leq 1$

(2) $\exists i,j \; : \; \| z(i,j) \|_2^2 \geq \dfrac{n^k}{N(n)}$

Proof of (1): Set $\Theta(x) = \sum_i x_i A_i \rightsquigarrow \Theta(x)^T \Theta(x) = \|x\|_2^2 \, \mathbb{1}_{N(n)}$

$$\langle x_1 \otimes \dots \otimes x_k \mid z(i,j) \rangle$$

$$= [\Theta(x_1) \cdots \Theta(x_k)]_{ij}$$

Now note that

$$[\Theta(x_1) \cdots \Theta(x_k)]_{ij}^2 = \langle i \mid \Theta(x_k)^T \cdots \Theta(x_1)^T \mid j \rangle\langle j \mid \Theta(x_1) \cdots \Theta(x_k) \mid i \rangle$$

$$\leq \langle i \mid \Theta(x_k)^T \cdots \underbrace{\Theta(x_1)^T \Theta(x_1)}_{= \|x_1\|_2^2 \mathbb{1}_{N(n)}} \cdots \Theta(x_k) \mid i \rangle$$

$$= \|x_1\|_2^2 \cdots \|x_k\|_2^2 .$$

$\square$

Finally, set $z = z(i,j)$ s.th. $\begin{cases} \|z\|_{\varepsilon_k(X)} \leq 1 \\ \|z\|_2^2 \geq \dfrac{n^k}{N(n)} \end{cases}$

$$\|z\|_2^2 \underset{\text{Duality of norms}}{\leq} \|z\|_{\varepsilon_k(X)} \|z\|_{\pi_k(X)} \leq \|z\|_{\pi_k(X)}$$

$$\tau_\infty(id_X) \geq \lim_{k\to\infty} \left(\frac{\|z\|_{\pi_k(X)}}{\|z\|_{\varepsilon_k(X)}} \right)^{\frac{1}{k}} \geq \lim_{k\to\infty} \frac{n}{N(n)^{\frac{1}{k}}} = n.$$

$\square$

Shows that $\tau_\infty(id_X) = n$.

2.) $T : X \to X$ arbitrary

We have $\tau_\infty(T) \leq \|T\|_{N(X \to X)}$

<u>Show:</u> $\tau_\infty(T) \geq \|T\|_{N(X\to X)}$

Duality $\Longrightarrow$ $\exists\, Q : X \to X$ s.th. $\|Q\|_{X\to X} = 1$ and

$$\|T\|_{N(X\to X)} = \mathrm{Tr}[Q \circ T]$$

Set $R := \int_{O(X)} U^{-1} \circ Q \circ T \circ U \, dU$

$$= \frac{\mathrm{Tr}[Q\circ T]}{n}\, id_X = \frac{\|T\|_{N(X\to X)}}{n}\, id_X$$

$$\Longrightarrow \quad \tau_\infty(R) \leq \tau_\infty(Q \circ T) \leq \tau_\infty(T)$$

$$\tau_\infty(R) = \frac{\|T\|_{N(X\to X)}}{n}\,\tau_\infty(id_X) = \|T\|_{N(X\to X)}.$$

$\square$

Further properties of tensor radii:

Setting $\varrho_\infty(X) = \tau_\infty(id_X)$ we have:

- $\sqrt{n} \leq \varrho_\infty(X) \leq n$
- If X has enough symmetries, then

$$\varrho_\infty(X) = \frac{n}{d(X, \ell_2^n)}$$

Banach-Mazur distance

$$d(X,Y) = \inf\{ \|u\|_{X\to Y} \|u^{-1}\|_{Y\to X} : u \text{ invertible} \}$$

- NTP for pairs (X,Y) still open!

Thank you

for your attention.

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