New techniques for bounding stabilizer rank

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Random tensors at CIRM 2022
March 18, 2022

arXiv:2110.07781
Computational basis

- Let $\{ |0\rangle, |1\rangle \} \subseteq \mathbb{C}^2$ be the computational basis for $\mathbb{C}^2$

  \[
  |0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
  \]

- Let $\{ |x\rangle : x \in \mathbb{F}_2^n \} \subseteq (\mathbb{C}^2)^\otimes n$ be the computational basis for $(\mathbb{C}^2)^\otimes n$

  \[
  |00\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad |01\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad |10\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad |11\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}
  \]
A state is a unit vector in $(\mathbb{C}^2)^\otimes n$ (mod phase, i.e. an element of $\mathbb{P}^{2^n-1}$).

We often omit normalization.

States in $\mathbb{C}^2$ are called qubits.

States are denoted $|\psi\rangle, |\phi\rangle$, etc.

$\langle\psi|$ denotes conjugate-transpose of $|\psi\rangle$.
Quantum circuits

General framework:
1. Prepare a computational basis state $|0 \cdots 0\rangle \in (\mathbb{C}^2)^\otimes n$.
2. Apply a unitary matrix $U|0 \cdots 0\rangle$
3. Measure in the computational basis. For $x \in \mathbb{F}_2^n$, $p(x) = |\langle x | U | 0 \cdots 0 \rangle|^2$. 
Quantum circuits

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This talk: Classical simulation of Clifford+T circuits via stabilizer rank
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Classical simulation of quantum circuits

**Question:** Given a classical description of a quantum circuit

Can it be simulated efficiently on a classical computer?
Types of simulation

• **Strong simulation:**
  Compute \( p(x) \) for all \( x \in \mathbb{F}_2^k \).

• **\( \epsilon \)-Strong simulation:**
  Find a probability vector \( \tilde{p} \) such that
  \[
  (1 - \epsilon)p(x) \leq \tilde{p}(x) \leq (1 + \epsilon)p(x) \quad \text{for all } x \in \mathbb{F}_2^k.
  \]

• **Weak simulation:**
  Sample elements of \( x \in \mathbb{F}_2^k \) from a probability distribution \( \tilde{p} \) such that
  \[
  \|\tilde{p} - p\|_1 \leq \epsilon
  \]
This talk: Classical simulation of Clifford+T circuits via stabilizer rank
Clifford circuits

The Clifford group is the group of unitaries $U: (\mathbb{C}^2)^\otimes n \rightarrow (\mathbb{C}^2)^\otimes n$ composed of Clifford gates

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \quad CNOT = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

[Gottesman-Knill 98]: Clifford circuits can be efficiently simulated.
Clifford+T circuits

The Clifford+T group is the unitary group $U: (\mathbb{C}^2)^\otimes n \rightarrow (\mathbb{C}^2)^\otimes n$ composed of Clifford gates

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad \text{CNOT} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$ 

and T-gates

$$T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}.$$ 

Motivation [Bravyi-Kitaev 05]: The Clifford+T group is “universal” for quantum computation ... i.e. its closure is all unitaries.
Classical simulation of Clifford+T circuits

**Question:** Given a classical description of a Clifford+T circuit

Can it be simulated efficiently on a classical computer?
Classical simulation of Clifford+T circuits

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Can it be simulated efficiently on a classical computer?
This talk: Classical simulation of Clifford+T circuits via stabilizer rank
Partial answer: Stabilizer rank

• A state $|\phi\rangle \in (\mathbb{C}^2)^\otimes n$ is a **stabilizer state** if $|\phi\rangle = U|0\rangle^\otimes n$ for some Clifford circuit $U$.

• The **stabilizer rank** of a state $|\psi\rangle \in (\mathbb{C}^2)^\otimes n$, denoted $\chi(|\psi\rangle)$, is the minimum number $r$ for which

$$|\psi\rangle = \sum_{i=1}^{r} c_i |\phi_i\rangle$$

for some $c_i \in \mathbb{C}$ and $|\phi_i\rangle$ stabilizer states.

• The $\delta$-approximate stabilizer rank of $|\psi\rangle$ is

$$\chi_\delta(|\psi\rangle) = \min\{\chi(|\mu\rangle): |||\psi\rangle - |\mu\rangle|| \leq \delta\}.$$
[Bravyi-Smith-Smolin 16, Bravyi-Gosset 16]: A Clifford+T circuit with $n$ T-gates can be simulated...

- **Strongly** with cost quadratic in $\chi(|H\rangle \otimes^n)$.
  Compute $p(x)$ for all $x \in \mathbb{F}_2^k$

- **$\epsilon$-Strongly** with cost linear in $\chi(|H\rangle \otimes^n)$.
  Find a probability vector $\tilde{p}$ such that $(1 - \epsilon)p(x) \leq \tilde{p}(x) \leq (1 + \epsilon)p(x)$

- **Weakly** with cost linear in $\chi_\delta(|H\rangle \otimes^n)$.
  Sample elements of $x \in \mathbb{F}_2^k$ from a probability distribution $\tilde{p}$ such that $\|\tilde{p} - p\|_1 \leq \epsilon$
Proof idea: \[\text{Clifford circuits can be efficiently simulated.}\]

\[T = S^\dagger H S|0\rangle \langle 1|\]

\[\text{Cliff+ n T}\]
Proof idea \[\text{[GK98]: Clifford circuits can be efficiently simulated.}\]

Let \( r = \chi(|H\rangle \otimes^n) \) and \(|H\rangle \otimes^n = \sum_{i=1}^r c_i |\phi_i\rangle \).

\(|\psi\rangle = \tilde{U}(|0\rangle \otimes |H\rangle \otimes^n) = \sum_{i=1}^r c_i \tilde{U}(|0\rangle \otimes |\phi_i\rangle) |0\rangle \otimes |H\rangle \otimes^n \]

By [GK98], can simulate measurements on each efficiently.
Known bounds on stabilizer rank

\( \chi(|H)^\otimes n) \)

- [Huang-Newman-Szegedy 20]: \( \chi(|H)^\otimes n) \) super-polynomial unless P=NP.
- [Bravyi-Smith-Smolin 16]: \( \chi(|H)^\otimes n) \geq \Omega(\sqrt{n}) \).
- [Peleg-Shpilka-Volk 21]: \( \chi(|H)^\otimes n) \geq \Omega(n) \).
- [Qassim-Pashayan-Gosset 21]: \( \chi(|H)^\otimes n) \leq 2^{\alpha n} \), where \( \alpha = \frac{1}{4} \log_2(3) \).

\( \chi_\delta(|H)^\otimes n) \)

- [Peleg-Shpilka-Volk 21]: There exists \( \delta > 0 \) such that \( \chi_\delta(|H)^\otimes n) \geq \Omega(\sqrt{n}/\log n) \).
- [Bravyi-Gosset 16]: \( \chi_\delta(|H)^\otimes n) \leq O\left(\frac{1}{\delta^2} 2^{\alpha m}\right) \), where \( \alpha \approx 0.228 \).

This talk: Alternate proofs up to log factor
Rest of talk

Lower bounds
- \( \chi(|H\otimes^n) \geq \Omega(n/\log n) \).
- There exists \( \delta > 0 \) such that \( \chi_\delta (|H\otimes^n) \geq \Omega(\sqrt{n}/\log n) \).

Match [Peleg, Shpilka, Volk 22] up to log factor

Upper bounds
- Generic stabilizer rank

Fact: If $|\phi\rangle \in (\mathbb{C}^2)^\otimes n$ is a stabilizer state, then the coordinates of $|\phi\rangle$ are \{0, ±1, ±i\} (up to normalization).
Theorem [Dehaene, De Moor 03]:

\[ |\phi\rangle \in (\mathbb{C}^2)^{\otimes n} \text{ is a stabilizer state } \iff |\phi\rangle = \sum_{x \in A} i^l(x)(-1)^q(x) |x\rangle, \]

where \( A \subseteq \mathbb{F}_2^n \) is an affine linear subspace

\( l: \mathbb{F}_2^n \to \mathbb{F}_2 \) is a linear function

\( q: \mathbb{F}_2^n \to \mathbb{F}_2 \) is a quadratic polynomial
Subset-sum representations

- Let \( \alpha \in \mathbb{C}^k, \beta \in \mathbb{C}^r \). We say \( \beta \) is a **subset-sum representation** of \( \alpha \) if each \( \alpha_j \) is equal to the sum of some subset of \( \{ \beta_1, \ldots, \beta_r \} \).

- **Example:** \( \beta = (1, 2) \) is a subset-sum representation of \( \alpha = (1, 2, 3) \).

- **Example:** If \( |\psi\rangle = \sum_{i=1}^{r} c_i |\phi_i\rangle \) is a stabilizer decomposition, then
  \[
  \beta = (c_1, \ldots, c_r, -c_1, \ldots, -c_r, \text{i}c_1, \ldots, \text{i}c_r, -\text{i}c_1, \ldots, -\text{i}c_r) \in \mathbb{C}^{4r}
  \]
  is a subset-sum representation of \( |\psi\rangle \).

  \( |\phi_i\rangle \) stabilizer \( \Rightarrow \) coordinates are in \( \{0, \pm 1, \pm \text{i} \} \)

\[\Rightarrow \chi(|\psi\rangle) \geq \frac{1}{4} \cdot (\text{the size of the smallest subset-sum rep of } |\psi\rangle)\]
Let \( \alpha \in \mathbb{C}^k, \beta \in \mathbb{C}^r \). We say \( \beta \) is a 
subset-sum representation of \( \alpha \) if each \( \alpha_j \) is equal to the sum of some subset of \( \{\beta_1, \ldots, \beta_r\} \).

Trivially, \( r \geq \log_2 k \), since \( \{\beta_1, \ldots, \beta_r\} \) has just \( 2^r \) different subsets.

Theorem [Moulton 01]: If \( 2 |\alpha_j| \leq |\alpha_{j+1}| \) for all \( j \in \{1, \ldots, k - 1\} \), then \( r \geq k / \log_2 k \).

Example: If \( \alpha = (2^1, 2^2, \ldots, 2^k) \), then \( r \geq k / \log_2 k \).
Lower bounds on the size of a subset-sum rep

• Let $\alpha \in \mathbb{C}^k, \beta \in \mathbb{C}^r$. We say $\beta$ is a subset-sum representation of $\alpha$ if each $\alpha_j$ is equal to the sum of some subset of $\{\beta_1, ..., \beta_r\}$.

• Trivially, $r \geq \log_2 k$, since $\{\beta_1, ..., \beta_r\}$ has just $2^r$ different subsets.

• **Theorem [Moulton 01]**: If $2|\alpha_j| \leq |\alpha_{j+1}|$ for all $j \in \{1, ..., k - 1\}$, then $r \geq k / \log_2 k$.

• **Example**: If $\alpha = (2^1, 2^2, ..., 2^k)$, then $r \geq k / \log_2 k$.

• **Theorem [Lovitz-Steffan]**: If the coordinates of $|\psi\rangle$ contain an exponentially increasing sequence of length $k$, then $\chi(|\psi\rangle) \geq k/(4\log_2 k)$.
Lower bound on stabilizer rank

- **Theorem [Lovitz-Steffan]**: If the coordinates of $|\psi\rangle$ contain an exponentially increasing sequence of length $k$, then $\chi(|\psi\rangle) \geq \frac{k}{4\log_2 k}$.

**Corollary [Lovitz-Steffan]**: $\chi(|H\rangle^{\otimes n}) \geq \frac{n}{4 \log_2 n}$.

**Proof:** Since $|H\rangle \approx |0\rangle + 2.41|1\rangle$,

$|H\rangle^{\otimes n} \approx |0\cdots 0\rangle + (2.41)(|0\cdots 01\rangle + \cdots + |10\cdots 0\rangle) + \cdots + (2.41)^n|1\cdots 1\rangle$.

$\Rightarrow |H\rangle^{\otimes n}$ contains the exponentially increasing sequence $(2.41, 2.41^2, \ldots, 2.41^n)$

$\Rightarrow \chi(|H\rangle^{\otimes n}) \geq \frac{n}{4 \log_2 n}$ by boxed theorem.
Lower bound on approximate stabilizer rank

- The $\delta$-approximate stabilizer rank of a normalized state $|\psi\rangle$ is
  \[ \chi_\delta(|\psi\rangle) = \min\{\chi(|\mu\rangle): \||\psi\rangle - |\mu\rangle\| \leq \delta\}. \]

- **Theorem [Lovitz-Steffan]:** There exists $\delta > 0$ for which
  \[ \chi_\delta(|H\rangle^{\otimes n}) \geq \sqrt{n}/(4 \log_2 \sqrt{n}). \]

  **Proof sketch:** Show that for $\delta$ small enough, any state that is $\delta$-close to $|H\rangle^{\otimes n}$ must contain an exponentially increasing sequence of length $\sqrt{n}$ (Use De Moivre-Laplace).

  Result follows from boxed theorem.
Super-linear lower bound on $\chi(|H\rangle^{\otimes n})$?

[BSS16] idea:
The T-count of a state $|\psi\rangle$ is the minimum number $n$ of T gates needed to prepare $|\psi\rangle$ with a Cliff+T circuit $U$

Fact: If $|\psi\rangle$ has T-count $n$, then $\chi(|\psi\rangle) \leq \chi(|H\rangle^{\otimes n})$.

Proof:

$$
\begin{align*}
T & = \begin{array}{c}
H \quad S^\dagger \quad |1\rangle\langle 1|
\end{array} \\
|H\rangle & \rightarrow \begin{array}{c}
\text{Cliff} \quad \text{T}
\end{array} \rightarrow |0\rangle \otimes |H\rangle^{\otimes n} \rightarrow U \rightarrow |\psi\rangle
\end{align*}
$$
Super-linear lower bound on $\chi(|H\rangle^\otimes n)$?

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The T-count of a state $|\psi\rangle$ is the minimum number $n$ of T gates needed to prepare $|\psi\rangle$ with a Cliff+T circuit $U$.

Fact: If $|\psi\rangle$ has T-count $n$, then $\chi(|\psi\rangle) \leq \chi(|H\rangle^\otimes n)$.

Proof:
Let $r = \chi(|H\rangle^\otimes n)$ and $|H\rangle^\otimes n = \sum_{i=1}^{r} c_i |\phi_i\rangle$.

$|\psi\rangle = \tilde{U}(|0\rangle \otimes |H\rangle^\otimes n) = \sum_{i=1}^{r} c_i \tilde{U}(|0\rangle \otimes |\phi_i\rangle)$

... so $\chi(|\psi\rangle) \leq r$. 

Stabilizer state!
Super-linear lower bound on $\chi(|H\rangle^\otimes n)$?

**Fact:** If $|\psi\rangle$ has T-count $n$, then $\chi(|\psi\rangle) \leq \chi(|H\rangle^\otimes n)$.

[BSS16] idea: For each $n$, if there exists a state $|\psi_n\rangle$ that:

1. Has T-count $n$
2. Satisfies $\chi(|\psi_n\rangle) \geq n^{1+\epsilon}$

... then $\chi(|H\rangle^\otimes n) \geq \chi(|\psi_n\rangle) \geq n^{1+\epsilon}$  
^1  
^2  

Super-linear
Super-linear lower bound on $\chi(|H\otimes n\rangle)$?

**Fact:** If $|\psi\rangle$ has T-count $n$, then $\chi(|\psi\rangle) \leq \chi(|H\otimes n\rangle)$.

**[BSS16] idea:** For each $n$, if there exists a state $|\psi_n\rangle$ that:

1. Has T-count $n$
2. Every subset-sum rep of $|\psi_n\rangle$ has size at least $n^{1+\epsilon}$

... then $\chi(|H\otimes n\rangle) \geq \chi(|\psi_n\rangle) \geq \frac{1}{4}n^{1+\epsilon}$ \hspace{1cm} Super-linear

**[Beverland-Campbell-Howard-Kliuchnikov 2020]:** A state of T-count $n$ can have an exponentially increasing sequence of length at most $O(n)$.
Rest of talk

Lower bounds
• $\chi(|H\otimes^n) \geq \Omega(n/ \log n)$.
• There exists $\delta > 0$ such that $\chi_\delta (|H\otimes^n) \geq \Omega(\sqrt{n}/ \log n)$.

Upper bounds
• Generic stabilizer rank

Match [Peleg, Shpilka, Volk 22] up to log factor

Upper bounds: Generic stabilizer rank

• Let \( \chi_n = \max\{\chi(|\psi\rangle^\otimes n): |\psi\rangle \in \mathbb{C}^2\} \) be the \( n \)-th generic stabilizer rank.

• \( \chi_n \geq \chi(|H\rangle^\otimes n) \)

• Fact: \( \chi(|\psi\rangle^\otimes n) = \chi_n \) for all but finitely many \( |\psi\rangle \in \mathbb{C}^2 \) (up to scale).

• Proposition [Lovitz-Steffan]: \( \chi_n = O\left(2^{n/2}\right) \)
  (Slight improvement of recent bound \( O((n + 1)2^{n/2}) \) of [Qassim-Pashayan-Gosset 21])

• Proposition [Lovitz-Steffan]: There exists a single set of \( \chi_n \) stabilizer states that can be superimposed to produce any state of the form \( |\psi\rangle^\otimes n \).
Summary

Classical simulation of Clifford+T circuits via stabilizer rank

Lower bounds
• $\chi(|H|^n) \geq \Omega(n/\log n)$.
• There exists $\delta > 0$ such that $\chi_\delta(|H|^n) \geq \Omega(\sqrt{n}/\log n)$.

Upper bounds
• Generic stabilizer rank

Better bounds?

Match [Peleg, Shpilka, Volk 22] up to log factor
New techniques for bounding stabilizer rank

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