Introduction to the Geometry of Tensors Part 1:

The fundamental theorem of linear algebra is a pathology + introduction to symmetry

J.M. Landsberg

Texas A&M University and Univ. Toulouse (chaire d'excellence)

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Overview

1. Linear algebra review
2. Tensors
3. First open questions
4. The fundamental theorem of linear algebra is a pathology
5. Why algebraic geometry (polynomials)?
6. Why representation theory (exploitation of symmetry)?
7. Asymptotic geometry of tensors (quantum information theory and complexity of matrix multiplication)

Notation

$A = \mathbb{C}^{\mathbf{a}}$: column vectors,

A^*: row vectors = space of linear maps $A \to \mathbb{C}$, where $\alpha \in A^*$, $v \in A$, $\alpha(v) = \alpha v$, row-column mult.

$\mathrm{End}(A) = \{\text{linear maps } A \to A\} \cong A^* \otimes A$

$GL(A)$ *group* of invertible linear maps $A \to A$
$= \{g \in \mathrm{End}(A) \mid \det(g) \neq 0\}$.

Similarly $B = \mathbb{C}^{\mathbf{b}}$, $C = \mathbb{C}^{\mathbf{c}}$.

Bilinear forms on $A^* \times B^*$

$M \in A \otimes B$ bilinear form i.e., $M : A^* \times B^* \to \mathbb{C}$. if choose bases $\mathbf{a} \times \mathbf{b}$ matrix

May also view as $M_A : A^* \to B$

$M_B : B^* \to A$

$GL(A) \times GL(B) \cdot M$ *orbit* of M.

$\overline{GL(A) \times GL(B) \cdot M}$ *orbit closure* of M.

Quiz: let M be "random", what is $\overline{GL(A) \times GL(B) \cdot M}$?

Normal forms: $\begin{pmatrix} \mathrm{Id}_r & 0 \\ 0 & 0 \end{pmatrix}$, $1 \leq r \leq \mathbf{a}$.

A bilinear form M is determined up to isomorphism by its rank.

In particular, rank one if $\exists a \in A,\ b \in B,\ M = a \otimes b$, i.e. column vect. $\times$ row vect.

Group actions

Bilinear forms: $GL(A) \times GL(B)$ acts on $A \otimes B$, finite number of orbits, simple normal form for each.

Use: efficient algorithm to solve system of linear equations (ancient China, rediscovered by Gauss) Exploit (part of) group action to put system in easy form.

Endomorphisms $A \to A$ v. Bilinear forms $A \times A \to \mathbb{C}$

$A^* \otimes A^*$: bilinear forms $A \times A \to \mathbb{C}$.

$GL(A)$ acts on $\mathrm{End}(A) = A^* \otimes A$. $g \in GL(A)$, $M \in A^* \otimes A$, $g \cdot M = gMg^{-1}$. Jordan normal form: infinite number of orbits (open subset described by **a** parameters) "tame" orbit structure.

Bilinear forms: $GL(A)$ acts on $A^* \otimes A^*$ $g \in GL(A)$, $M \in A^* \otimes A^*$, $g \cdot M = gMg^t$. Normal form? In general, no but see Conner-Gesmundo-L-Ventura

Fundamental Theorem of linear algebra

Fix bases $\{a_i\}$, $\{b_j\}$ of A, B and for $r \leq \min\{\mathbf{a}, \mathbf{b}\}$, set $I_r = \sum_{k=1}^r a_k \otimes b_k$.

The following quantities all equal the **rank** of $T \in A \otimes B$:

(Q) The largest r such that $I_r \in \mathrm{End}(A) \times \mathrm{End}(B) \cdot T$.

($\underline{\mathbf{Q}}$) The largest r such that $I_r \in \overline{GL(A) \times GL(B) \cdot T}$.

($\mathbf{ml}_A$) $\dim A - \dim \ker(T_A : A^* \to B)$

($\mathbf{ml}_B$) $\dim B - \dim \ker(T_B : B^* \to A)$

($\underline{\mathbf{R}}$) The smallest r such that T is a limit of a sum of r rank one elements, i.e., such that $T \in \overline{GL(A) \times GL(B) \cdot I_r}$

(R) The smallest r such that T is a sum of r rank one elements. i.e., such that $T \in \mathrm{End}(A) \times \mathrm{End}(B) \cdot I_r$

Tensors

Now consider trilinear form $A^* \times B^* \times C^* \to \mathbb{C}$.

if choose bases $\mathbf{a} \times \mathbf{b} \times \mathbf{c}$ array

$T \in A \otimes B \otimes C$. (or $T \in A_1 \otimes \cdots \otimes A_k$)

Bilinear map $A^* \times B^* \to C$.

Linear map $T_A : A^* \to B \otimes C$

Example: $A^*, B^*, C = \mathcal{A}$ algebra, $T = T_{\mathcal{A}}$ structure tensor. i.e., $T_{\mathcal{A}}(a_1, a_2) := a_1 a_2$.

In particular, A, B, C space of $n \times n$ matrices $T = M_{\langle n \rangle}$ structure tensor of matrix multiplication.

$T \in A \otimes B \otimes C$ has *rank one* if $\exists$ $a \in A$, $b \in B$, $c \in C$ such that $T = a \otimes b \otimes c$.

Tensors

Can consider

$GL(A) \times GL(B) \times GL(C) \cdot T$ *orbit* of T.

$\overline{GL(A) \times GL(B) \times GL(C) \cdot T}$ *orbit closure* of T.

Let T be "random", what is $\overline{GL(A) \times GL(B) \times GL(C) \cdot T}$?

too difficult, instead:

What is $\dim(\overline{GL(A) \times GL(B) \times GL(C) \cdot T})$?

Trick question Answer $\mathbf{a}^2 + \mathbf{b}^2 + \mathbf{c}^2 - 2$.

Note ambient space dimension **abc**

Choose inclusions $A \subset \mathbb{C}^s$, $B \subset \mathbb{C}^s$, $C \subset \mathbb{C}^s$, (think of s as large)

$\rightsquigarrow A \otimes B \otimes C \subset \mathbb{C}^s \otimes \mathbb{C}^s \otimes \mathbb{C}^s$.

$\mathbb{C}^s$ bases $\{e_\ell\}$, $\{f_\ell\}$, $\{g_\ell\}$

Write $I_r = \sum_{\ell=1}^r e_\ell \otimes f_\ell \otimes g_\ell$, $1 \leq r \leq s$.

Tensors

Definitions:

$\mathbf{Q}(T)$ *subrank*: largest r such that
$I_r \in \mathrm{End}(A) \times \mathrm{End}(B) \times \mathrm{End}(C) \cdot T$

$\underline{\mathbf{Q}}(T)$ *border subrank*: largest r such that
$I_r \in \overline{GL(A) \times GL(B) \times GL(C) \cdot T}$

ml *multi-linear ranks*:
$(\mathbf{ml}_A(T), \mathbf{ml}_B(T), \mathbf{ml}_C(T)) := (\mathrm{rank}\, T_A, \mathrm{rank}\, T_B, \mathrm{rank}\, T_C)$

$\underline{\mathbf{R}}(T)$ *border rank*: The smallest r such that T is a limit of rank r tensors i.e. such that $T \in \overline{GL(A) \times GL(B) \times GL(C) \cdot I_r}$

$\mathbf{R}(T)$ *rank*: smallest r such that T is a sum of r rank one tensors i.e., such that $T \in \mathrm{End}(A) \times \mathrm{End}(B) \times \mathrm{End}(C) \cdot I_r$.

Inequalities and first open problems

$$\mathbf{Q}(T) \leq \underline{\mathbf{Q}}(T) \leq \min\{\mathbf{ml}_A(T), \mathbf{ml}_B(T), \mathbf{ml}_C(T)\}$$
$$\leq \max\{\mathbf{ml}_A(T), \mathbf{ml}_B(T), \mathbf{ml}_C(T)\} \leq \underline{\mathbf{R}}(T) \leq \mathbf{R}(T)$$

all may be strict, even when $\mathbf{a} = \mathbf{b} = \mathbf{c}$.

Note $\mathbf{ml}_A(T) \leq \min\{\mathbf{a}, \mathbf{bc}\}$ etc.

For simplicity, say $\mathbf{a} = \mathbf{b} = \mathbf{c} = m$,

2021 Open: What is $\mathbf{Q}(T)$ for a random tensor?

2022 Derksen-Makam-Zuiddam Theorem (see Zuiddam's lecture for answer!)

Open: What is $\underline{\mathbf{Q}}(T)$ for a random tensor?

Rank and border rank

T: random $\Rightarrow \underline{\mathbf{R}}(T) = \mathbf{R}(T) \simeq \frac{m^2}{3}$ and this is largest possible $\underline{\mathbf{R}}$. (Lickteig 1985, symmetric case Terracini 1916, higher order symmetric mostly Terracini 1916, finished Alexander-Hirschowitz 1990's)

Unlike matrices, a random tensor will *not* have maximal rank!

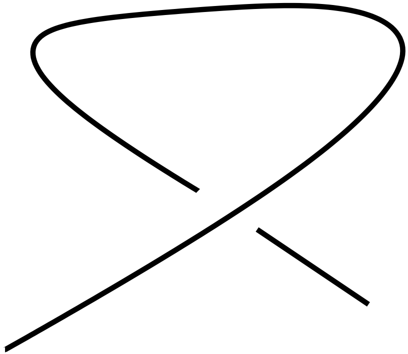
Open: Largest possible $\mathbf{R}(T)$? (state of art, see Buczynski-Han-Mella-Teitler)

Open: rank of a random tensor in $A_1 \otimes \cdots \otimes A_k$ (see Abo-Ottaviani-Peterson for state of art). Rems: have expected answer, known correct in many cases, will be equal to border rank of random tensor

If multilinear ranks maximal $= m$, call T *concise* $\Rightarrow \underline{\mathbf{R}}(T) \geq m$, say *minimal border rank* if $= m$.

Open: Classify concise tensors of minimal border rank. (state of art: March 2022 Jelisiejew-L-Pal $m \leq 5$)

Geometry of rank: pathology of fundamental theorem

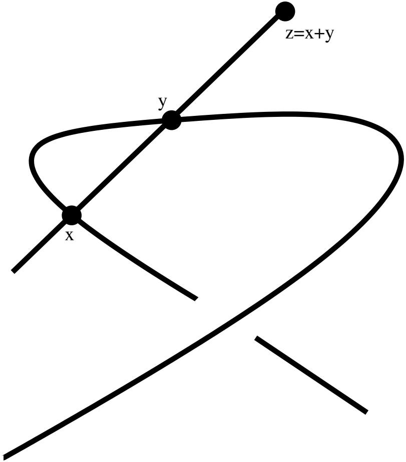


Imagine curve represents the set of tensors of rank one sitting in the N^3 dimensional space of tensors.

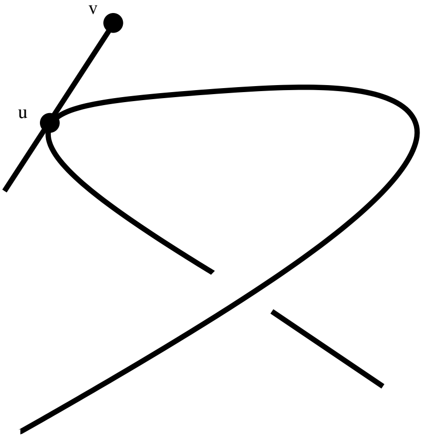
Geometry of rank

$\{$ tensors of rank two$\}$ $=$

$\{$ points on a secant line to set of tensors of rank one$\}$



Geometry of border rank



The limit of secant lines is a tangent line!

Note: most points on just one secant line.

Most points: if on secant line, usually not on tangent line

Plane curve: both. Rank one matrices like curves in the plane

Pathology!

Theorem (Zak 1980's/Severi 1910's): Rank one matrices and rank one symmetric matrices essentially only smooth (in projective space) geometric objects with rank semi-continuity.

Polynomials and limits

Clear: P: poly, $P(T_t) = 0$ for $t > 0 \Rightarrow P(T_0) = 0$.

$\Rightarrow$ Cannot describe tensor rank via zero sets of polynomials.

Matrices: Matrix border rank given by polynomials.

Tensor border rank?

Tensors of border rank $\leq r$ Euclidean closed

$S \subset V$ set, define *Zariski closure* by first define ideal
$I_S := \{\text{polys } P \mid P(s) = 0 \forall s \in S\}$.

$\overline{S}^{zar} := \{v \in V \mid P(v) = 0 \forall P \in I_S\}$.

Theorem (Mumford 1960's): In our situation $\overline{S} = \overline{S}^{zar}$ (whenever $\overline{S}^{zar}$ is irreducible and S contains a Zariski-open subset of $\overline{S}^{zar}$).

$\Rightarrow$ can determine border rank with polynomials!

Border rank via Polynomials

Matrices: easy, just minors (efficient to compute thanks to Gaussian elimination)

Tensors??

Open

State of the art: border rank ≤ 4 (Friedland 2010)

Next time: some known equations and motivation.

Normal forms?

Bilinear forms: finite number of orbits

Endomorphisms: finite number of cases, each with finite number of parameters "tame"

Tensors?

Kronecker $\mathbb{C}^2 \otimes \mathbb{C}^{\mathbf{a}} \otimes \mathbb{C}^{\mathbf{b}}$: yes! tame

$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3$: yes! tame

In general: NO "wild"

Tensor Rank Decomposition

Linear algebra: determine rank of matrix easy. finding a rank decomposition easy. $r > 1$, never unique.

Tensors: determine rank of tensor hard. No general technique. (methods for T low rank and with nice combinatorial properties)
But: often unique!

If can decompose, extremely useful for applications.

e.g. blind source separation (P. Comon)

Classical algebraic geometry

Consider rank at most r matrices:
$\sigma_r(Seg(\mathbb{P}A \times \mathbb{P}B)) = \{[T] \mid \underline{\mathbf{R}}(T) \leq r\}$

Invariant under changes of bases $\Rightarrow$ its ideal
$I_{\sigma_r(Seg(\mathbb{P}A\times\mathbb{P}B))} \subset Sym(A^*{\otimes}B^*)$ invariant under changes of bases

Special case: rank one - saw matrix has rank one iff size two minors zero. Degree two polynomials.

Consider all homogeneous degree two polynomials on matrices:

$S^2(A^*{\otimes}B^*) = S^2A^*{\otimes}S^2B^* \oplus \Lambda^2A^*{\otimes}\Lambda^2B^*$

Size two minors ??

What about $S^2(A^*{\otimes}B^*{\otimes}C^*)$? More generally any subspace in $I_{\sigma_r(Seg(\mathbb{P}A\times\mathbb{P}B\times\mathbb{P}C))}$?

Quantum information theory

- $T \in A \otimes B \otimes C$, $T' \in A' \otimes B' \otimes C'$, define *Kronecker product* $T \boxtimes T' \in (A \otimes A') \otimes (B \otimes B') \otimes (C \otimes C')$, and Kronecker powers $T^{\boxtimes k} \in (A^{\otimes k}) \otimes (B^{\otimes k}) \otimes (C^{\otimes k})$
- Say T *degenerates* to T' if $T' \in \overline{GL(A) \times GL(B) \times GL(C) \cdot T}$. In this case $\underline{\mathbf{R}}(T') \leq \underline{\mathbf{R}}(T)$.

Cost v. Value in quantum information: Approximate Cost of $T \sim \underline{\mathbf{R}}(T)$, Approximate Value $\sim \underline{\mathbf{Q}}(T)$,

True cost/value $\underset{\sim}{\mathbf{R}}(T) := \lim_{N \to \infty} (\underline{\mathbf{R}}(T^{\otimes N}))^{\frac{1}{N}}$,

$\underset{\sim}{\mathbf{Q}}(T) := \lim_{N \to \infty} (\underline{\mathbf{Q}}(T^{\otimes N}))^{\frac{1}{N}}$

Find low cost high value tensors. Exchange rate on Quantum information market (see Christandl lecture)

Approaches to value

$\mathbf{Q}$, $\underline{\mathbf{Q}}$ not related to classically studied objects.

Chang (2022): unlike rank and border rank, tensors with maximal $\underline{\mathbf{Q}}$ are vastly more abundant than tensors with maximal $\mathbf{Q}$: more precisely the dimensions of these sets differ greatly.

Idea: define easier to compute quantities bounding $\underline{\mathbf{Q}}$

$\rightsquigarrow$ slice rank (Tao, 2016) and Strength/product rank (for higher order tensors)

Approaches to value, cont'd

"If a polynomial/tensor is biased—in the sense that its output distribution deviates significantly from uniform—must it be the case that it is algebraically structured, in the sense that it is a function of a small number of lower-degree polynomials/tensors?"

Variant over finite fields inspired by random tensors: analytic rank (Gowers) "low (product) rank implies bias" Cohen-Moshkovitz (2021) : bias implies low (product) rank.

$\rightsquigarrow$ geometric rank (Kopparty-Moshkovitz-Zuiddam, 2020) over all fields

$\rightsquigarrow$ classical linear algebra *and* classical algebraic geometry:

spaces of matrices of bounded rank, linear $\mathbb{P}^{m-1}$'s $\subset \mathbb{P}(\mathbb{C}^m \otimes \mathbb{C}^m)$ having non-transverse interections with $\sigma_r(Seg(\mathbb{P}^{m-1} \times \mathbb{P}^{m-1}))$

L-Geng (2021): low geometric rank implies high tensor rank. Geng (2022): classification of geometric rank ≤ 3 and general results on geometry of tensors with low geometric rank.

Thank you for your attention

For more on **tensors, their geometry and applications**, resp. **geometry and complexity**, resp. **asymptotic geometry, moment maps, (quantum) information theory...** : :