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References

General decompositions with invariance, positivity and approximations

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Can we represent tensors explicitly invariant and locally positive?

Positivity

Tensor T is *nonnegative* if $T_{i_0,\ldots,i_n} \geq 0$ for all $\mathbf{i}$.

Invariance

$$T_{i_0,i_1,i_2} = T_{i_{\sigma(0)},i_{\sigma(1)},i_{\sigma(2)}}$$
for all $\sigma \in G \subseteq S_3$.

Every tensor decomposes into

$$T = \sum_{\alpha=1}^{r} v_\alpha^{[1]} \otimes v_\alpha^{[2]} \otimes v_\alpha^{[3]}$$

- *Invariant:* $v_\alpha^{[i]}$ equal for every i.
- *Positive:* $v_\alpha^{[i]}$ nonnegative vector.
- Different arrangements of multiple summation indices.

- When does an *invariant* and *positive* decomposition exist?
- Including local positivity, does the *rank* of the decomposition change?
- Are rank separations robust with respect to approximations?

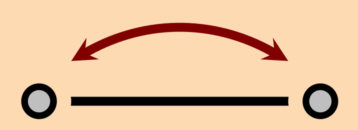
Existence of invariant decompositions

Example

Given $T \in \mathbb{C}^d \otimes \mathbb{C}^d$ such that

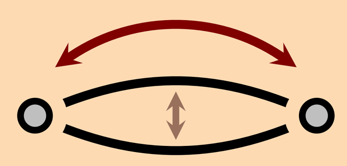
- $T_{i_0,i_1} = T_{i_1,i_0}$
- T is nonnegative

Decompose any such T reflecting *symmetry* **and** *positivity*



$$T = \sum_{\alpha=1}^{r} v_\alpha \otimes v_\alpha$$

✗



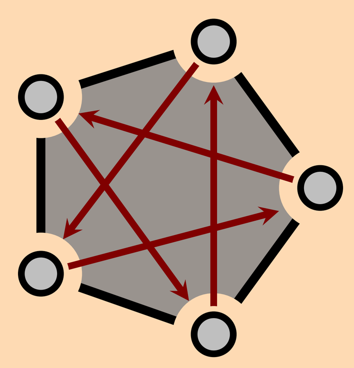
$$T = \sum_{\alpha,\beta=1}^{r} v_{\alpha,\beta} \otimes v_{\beta,\alpha}$$

✓

Main Theorem (informal)

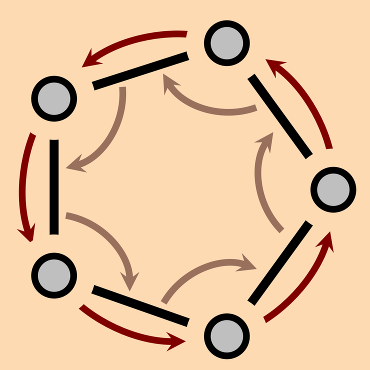
Every *invariant and nonnegative* tensor attains an *explicitly invariant, positive* decomposition when introducing additional summation indices.

Extension to other geometries [5]



Fully symmetric

$$T = \sum_{\alpha=1}^{r} v_\alpha \otimes v_\alpha \otimes \cdots \otimes v_\alpha$$



Translational invariant non.-neg. MPS

$$T = \sum_{\alpha_i=1}^{r} v_{\alpha_1,\alpha_2} \otimes v_{\alpha_2,\alpha_3} \otimes \cdots \otimes v_{\alpha_n,\alpha_1}$$

Open questions/Future work

- What happens to approximations when $p = 1$?
- Are there non-trivial border ranks for positive tensor decompositions?

  For example: $\lim_{n\to\infty} \operatorname{nn-rank}(W_n) \overset{?}{<} \operatorname{nn-rank}(T)$

  for a sequence $W_n \to T$

Rank separations

Cost

$$\operatorname{rank}(T) \ll \operatorname{nn-rank}(T)$$

There is **no** function $f : \mathbb{N} \to \mathbb{N}$ such that

$$\operatorname{nn-rank}(T) \leq f(\operatorname{rank}(T))$$

independent of the dimension [2,3].

The approximate case [6]

Let $\varepsilon > 0$ and $p > 1$.

For a nonnegative tensor T, there exists W with $\|T - W\|_{\ell_p} < \varepsilon$

such that
$$\operatorname{nn-rank}(W) \leq C \cdot \frac{\|T\|_{\ell_1}}{\varepsilon^{p/(p-1)}}$$

- Rank separation is not robust when fixing the error.

Main Tool

Approx. Carathéodory theorem for uniformly smooth Banach spaces [4].

Generalizations of this result

- Other decomposition geometries with invariance
- Other positive decompositions (e.g. psd-, sqrt-decomposition etc.)
- Other positivity cones:

  Tensor product space of positive semidefinite matrices

  Multivariate sum-of-squares/nonnegative polynomials [7]

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[6] G. De las Cuevas, A.K. , T. Netzer, *Approx. tensor dec.: Disappearance of many separations.* J. Math. Phys. **62** (2021)

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