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Entanglement Entropy of Pure Quantum States in Fermionic Many-Body Systems

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collaborators:

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Random Tensors at CIRM
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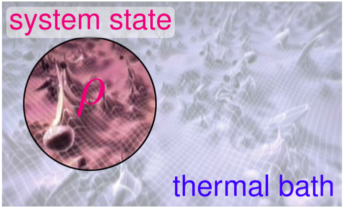
Entanglement Entropy

- System coupled to a thermal bath (equilibrium ensemble):

$$\rho = \frac{e^{-\hat{H}/T}}{Z} = \sum_i \underbrace{\frac{e^{-E_i/T}}{Z}}_{p_i} |E_i\rangle\langle E_i|$$



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(Thermal entropy)

images are by Lucas Hackl

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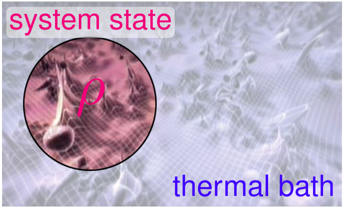
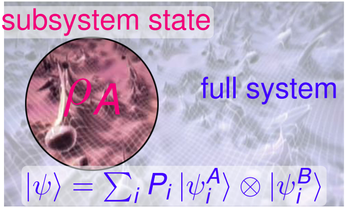


- Subsystem A in larger system (non-equilibrium situation):

$$\rho_A = \sum_i P_i |\psi_i^A\rangle\langle\psi_i^A| = \mathrm{Tr}_B \, |\psi\rangle\,\langle\psi|$$

$$S_A = -\sum_i P_i \ln P_i = -\mathrm{Tr}\,\rho_A \ln \rho_A$$

(Entanglement entropy)



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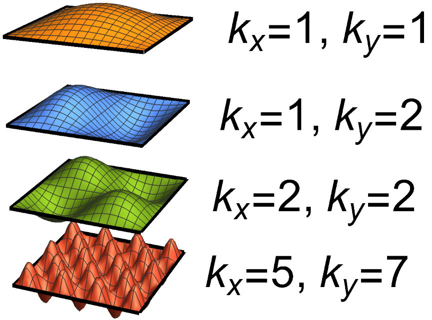
Integrable vs Chaotic

- Integrable systems like regular quantum billiards have regular wave functions.

⇒ Entanglement entropy cannot be maximal.

⇒ Thermalisation cannot happen for integrable systems.



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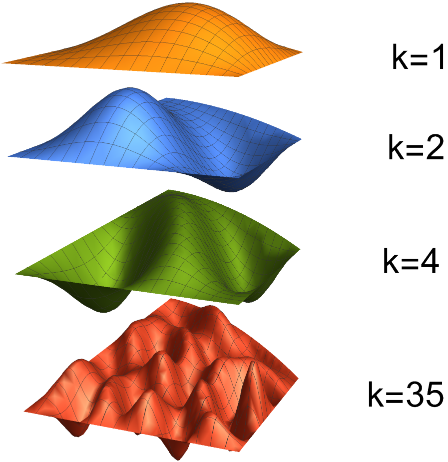

E.g., Porter-Thomas distribution for the distribution of eigenvector coefficients applies.

Eigenvectors look like Haar distributed unit vectors!

Page's Idea (based on Lubkin 78', Pagels, Lloyd 88')

- Eigenvectors of chaotic quantum system $|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ seem to be close to Haar distributed unit vectors.

- Let $\dim \mathcal{H}_A = d_A$ and $\dim \mathcal{H}_B = d_B$ implying $\dim \mathcal{H} = d = d_A d_B$.

- Choose a Haar distributed unit vector $|\psi\rangle \in S^{2d-1} = \mathrm{U}(d)/\mathrm{U}(d-1)$.

- **Page's idea (93'):** Consider the quantum state $\rho = |\psi\rangle\langle\psi|$ and the reduced density matrix of the subsystem A is $\rho_A = \mathrm{Tr}_B \rho$.

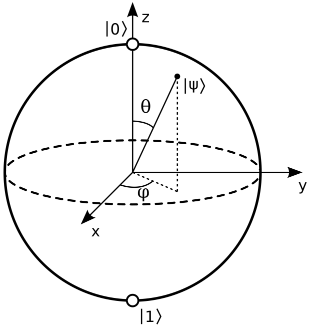

image from Wikipedia

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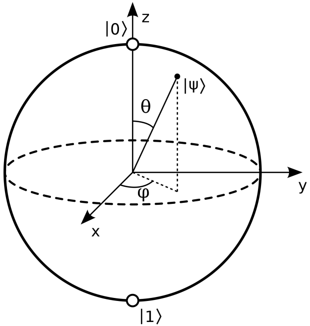

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Conjecture:
The average entanglement entropy $\langle S_A \rangle = \langle \mathrm{Tr}\rho_A \ln\rho_A \rangle$ is the generic one for an eigenstate of a complex quantum system when $d_A, d_B \to \infty$.

Pure States

- $|\psi\rangle \in S^{2d-1} \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be expanded:

$$|\psi\rangle = \sum_{a=1}^{d_A} \sum_{b=1}^{d_B} W_{ab} |a\rangle \otimes |b\rangle ,$$

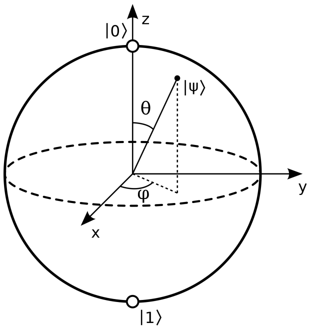
  where $\{|a\rangle\} \subset \mathcal{H}_A$ and $\{|b\rangle\} \subset \mathcal{H}_B$ are orthonormal bases.

- Collect coefficients in terms of a matrix $W = \{W_{ab}\} \in \mathbb{C}^{d_A \times d_B}$.
- **Normalisation:** $||\psi|| = 1$ reads as follows

$$1 = \langle\psi|\psi\rangle = \mathrm{Tr} W^\dagger W.$$

- **Reduced density matrix** is given as

$$\rho_A = \mathrm{Tr}_B |\psi\rangle \langle\psi| = WW^\dagger.$$



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Haar distributed Pure States

- Let $|\psi\rangle \in S^{2d-1} \in \mathcal{H}_A \otimes \mathcal{H}_B$ be Haar distributed (uniformly distributed):

$$|\psi\rangle = \sum_{a=1}^{d_A} \sum_{b=1}^{d_B} W_{ab} \, |a\rangle \otimes |b\rangle \, .$$

- **Random Matrix Ensemble:** W is distributed by

$$P(W) = \frac{\delta(1 - \mathrm{Tr}\, WW^\dagger)}{\int \delta(1 - \mathrm{Tr}\, WW^\dagger) d[W]} .$$

This is a fixed trace ensemble!

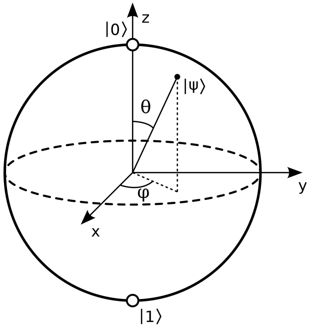


image from Wikipedia

Idea: Tracing this ensemble back to the complex Wishart-Laguerre ensemble.

Historical Results

Average entanglement entropy:

$$\langle S_A \rangle = \overbrace{\Psi(d_A d_B + 1)}^{\text{normalisation}} - \Psi(d_B + 1) - \frac{d_A - 1}{2d_B} = \ln(d_A) - \frac{d_A}{2d_B} + o(1)$$

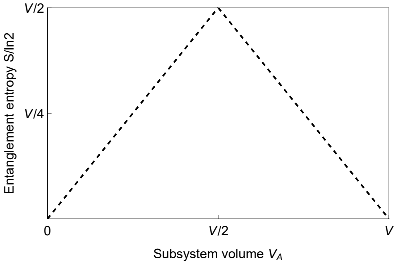
Digamma function: $\Psi(x) = \partial \ln \Gamma(x)$

For a qubit system:

1. $d_A = 2^{V_A}$, $d_B = 2^{V_B}$
2. $V = V_A + V_B \to \infty$
3. $\lim_{V\to\infty} V_A/V = f \in (0, 1/2]$

$$\langle S_A \rangle \longrightarrow \ln(2) f V - 2^{-(1-2f)V-1}$$

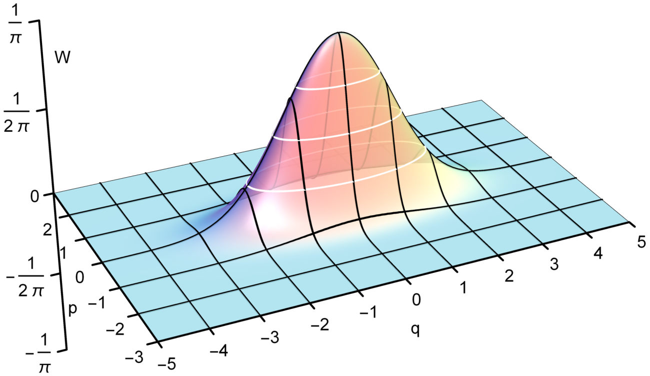
Page Curve:



Second term is only present when $1-2f \propto 1/V$.

Page (93'), Foong, Kanno (94'); Sánchez-Rui (95'); Sen (96')

Pure Fermionic Gaussian States



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What is a Fermionic Gaussian State?

Definition: A fermionic Gaussian state is

$$\rho = \exp[-\gamma Q \gamma^\dagger], \text{ with } Q = -Q^T \in i\mathbb{R}^{2V\times 2V}$$

and $\gamma = (\gamma_1, \ldots, \gamma_{2V})$ are Majorana fermions meaning they build a Clifford algebra in the irreducible matrix representation in $\mathbb{C}^{2^V\times 2^V}$

$$\gamma_a\gamma_b + \gamma_b\gamma_a = \delta_{ab}\mathbb{1}_{2^V}.$$

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Let $\lambda_j \geq 0$ be the singular values of Q and $\{\eta_j\} \subset \text{span}\{\gamma_a\}$ the corresponding eigenbases of the Majorana fermions. Then,

$$\rho = \prod_{j=1}^{V} \frac{\mathbb{1}_{2^V} + 2i\tanh(\lambda_j)\eta_{2j-1}\eta_{2j}}{2}.$$

Correlation Matrix

- The **correlation matrix** (only antisymmetric part)

$$J_{ab} = -J_{ba} = i\,\mathrm{Tr}\left[\rho\left(\gamma_a\gamma_b - \frac{1}{2}\mathbb{1}_{2^V}\right)\right]$$

comprises all information of a Gaussian state.

Peschel (03'); Bianchi, Hackl (20')

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- The singular values of J are $x_1, \ldots, x_V \in [0, 1]$ and the von Neumann entropy is

$$S = \mathrm{Tr}(\rho \ln \rho) = \sum_{j=1}^{V} s(x_j) \qquad \text{with}$$

$$s(x) = \frac{1+x}{2}\ln\left(\frac{1+x}{2}\right) + \frac{1-x}{2}\ln\left(\frac{1-x}{2}\right)$$

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Pure Fermionic Gaussian State = all $x_j = 1$

Peschel (03'); Bianchi, Hackl (20')

Generic Pure Fermionic Gaussian State

- Group action on pure fermionic Gaussian states is given by

$$J \longrightarrow OJO^T \text{ with } O \in \mathrm{O}(2V)$$

- Due to $J^2 = \mathbb{1}_N$ and $J = -J^T = J^*$ all correlation matrices can be written as follows

$$J = O \overbrace{\begin{bmatrix} 0 & \mathbb{1}_V \\ -\mathbb{1}_V & 0 \end{bmatrix}}^{=J_0} O^T$$

- Subgroup satisfying $J_0 = OJ_0O^T$ is given by $O \in \mathrm{U}(V)$ in the real $2V \times 2V$ matrix representation.

$\Rightarrow$ Manifold of all pure fermionic Gaussian state: $\mathrm{O}(2V)/\mathrm{U}(V)$

- Choose a Haar distributed $O \in \mathrm{O}(2V)$ to create a uniformly distributed state.

Bianchi, Hackl (20')

Reduced Density Matrix

- System A is given by Majorana fermions $\gamma_1, \ldots, \gamma_{2V_A}$ and the reduced density matrix is still a Gaussian state, namely an embedded Random Matrix.

- Reduced correlation matrix J_A is an orthogonal rank V_A projection of J, namely (Bianchi, Hackl (20'))

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- The jpdf of the singular values of J_A for $f = V_A/V \in [0, 1/2]$ is (Bianchi, Hackl, Kieburg (21'))

$$p(x_1, \ldots, x_{V_A}) \propto \prod_{a<b} (x_b^2 - x_a^2)^2 \prod_{j=1}^{V} (1 - x_j^2)^{V-2V_A}$$

- There is still the subsystem-subsystem symmetry!

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Can be solved with Jacobi polynomials!

Average Entanglement Entropy

$$\langle S_A \rangle = \int d[x] p(x) \sum_{j=1}^{V} \left[\frac{1+x_j}{2} \ln\left(\frac{1+x_j}{2}\right) + \frac{1-x_j}{2} \ln\left(\frac{1-x_j}{2}\right) \right]$$

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Bianchi, Hackl, Kieburg (21'); leading order already found by Lydźba, Rigol, Vidmar (20')

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$$= \left(V - \frac{1}{2}\right)\Psi(2V) + \left(\frac{1}{2} + V_A - V\right)\Psi(2V - 2V_A)$$

$$+ \left(\frac{1}{4} - V_A\right)\Psi(V) - \frac{1}{4}\Psi(V - V_A) - V_A$$

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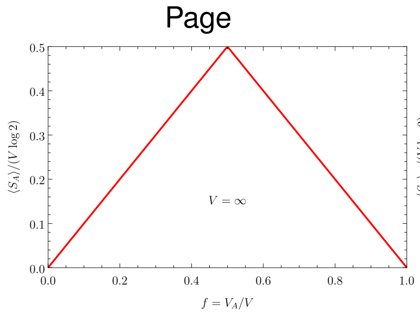
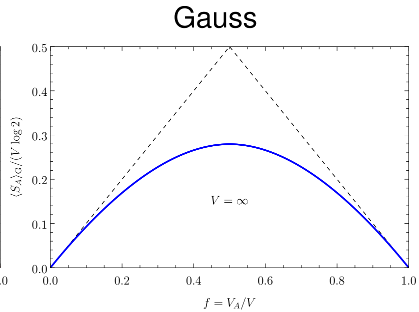
$$= V \left[(\ln(2) - 1) f + (f-1) \ln(1-f) \right] + \frac{f}{2} + \frac{\ln(1-f)}{4} + \mathcal{O}(V^{-1})$$

Contribution from the Page curve

Bianchi, Hackl, Kieburg (21'); leading order already found by Lydźba, Rigol, Vidmar (20')

Comparison





Reduced density matrices of Gaussian pure states are usually not maximally entangled!

Pure Gaussian Fermion States with Particle Preservation

- Go over to annihilation-creation operators

$$f_j = \frac{1}{\sqrt{2}}(\gamma_{2j-1} + i\gamma_{2j}) \qquad \text{and} \qquad f_j^\dagger = \frac{1}{\sqrt{2}}(\gamma_{2j-1} - i\gamma_{2j})$$

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- Respective correlation matrix is

$$\hat{J} = i \begin{bmatrix} \langle\psi| f_a f_b^\dagger - f_b^\dagger f_a |\psi\rangle & \langle\psi| f_a^\dagger f_b^\dagger - f_b^\dagger f_a^\dagger |\psi\rangle \\ \langle\psi| f_a f_b - f_b f_a |\psi\rangle & \langle\psi| f_a^\dagger f_b - f_b f_a^\dagger |\psi\rangle \end{bmatrix} \in \mathbb{C}^{2V \times 2V}$$

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- Number preserving pure state yields

$$\hat{J} = i \begin{bmatrix} \langle\psi| f_a f_b^\dagger - f_b^\dagger f_a |\psi\rangle & 0 \\ 0 & \langle\psi| f_a^\dagger f_b - f_b f_a^\dagger |\psi\rangle \end{bmatrix} = i \begin{bmatrix} F & 0 \\ 0 & -F^T \end{bmatrix}$$

with $F \in \mathrm{Herm}(V) \cap \mathrm{U}(V) \;\Rightarrow\; F^2 = \mathbb{1}_V$

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with $F \in \mathrm{Herm}(V) \cap \mathrm{U}(V) \;\Rightarrow\; F^2 = \mathbb{1}_V$

- Relation to particle number: $\mathrm{Tr}(F) = 2N - V$

Reduced Number Preserving States

- Manifold of all pure Gaussian states with exactly N fermions is given by $\mathrm{U}(V)/[\mathrm{U}(N) \times \mathrm{U}(V-N)]$. Elements are given by

$$F = U\mathrm{diag}(\mathbb{1}_N, -\mathbb{1}_{V-N})U^\dagger, \qquad U \in \mathrm{U}(V)$$

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- It is $\Pi F \Pi^T = 2 U_A U_A^\dagger - \mathbb{1}_{V_A}$ with $U_A \in \mathbb{C}^{V_A \times N}$ upper left block of $U \in \mathrm{U}(V)$.

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- Choose Haar a distributed $U\in\mathrm{U}(V)$ to create a uniformly distributed ensemble on $\mathrm{U}(V)/[\mathrm{U}(N)\times\mathrm{U}(V-N)]$.

- Reduced correlation matrix is given by

$$\hat{J}_A = i\,\mathrm{diag}(\Pi F\Pi^T, -\Pi F^T\Pi^T)$$

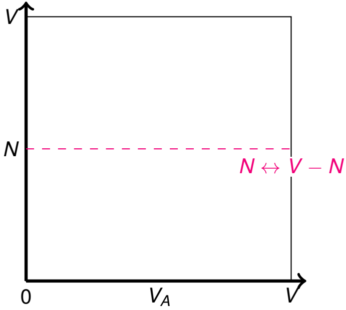
with Π is the orthogonal projection to first V_A rows.

- It is $\Pi F\Pi^T = 2U_AU_A^\dagger - \mathbb{1}_{V_A}$ with $U_A\in\mathbb{C}^{V_A\times N}$ upper left block of $U\in\mathrm{U}(V)$.

This is the complex Jacobi ensemble!

Symmetries

1. **Particle-hole symmetry: $N \leftrightarrow V - N$**



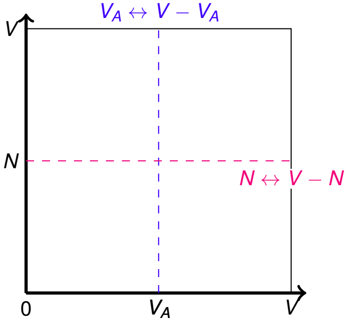
graphic courtesy by Lucas Hackl

Peschel (03'); Hackl, Bianchi (20')

Symmetries

1. **Particle-hole symmetry:** $N \leftrightarrow V - N$

2. **Subsystem-subsystem symmetry:** $V_A \leftrightarrow V - V_A$



graphic courtesy by Lucas Hackl

Peschel (03'); Hackl, Bianchi (20')

Symmetries

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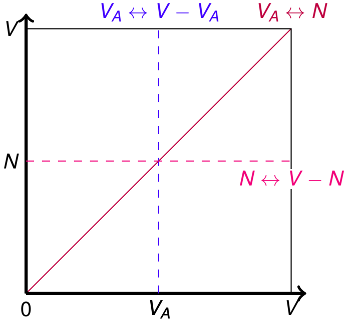
   $U_A U_A^\dagger \longleftrightarrow U_A^\dagger U_A$



graphic courtesy by Lucas Hackl

Peschel (03'); Hackl, Bianchi (20')

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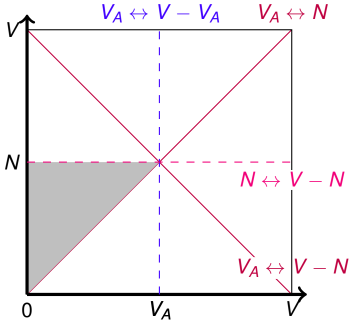

graphic courtesy by Lucas Hackl

$\Rightarrow$ It suffices to compute $\langle S_A \rangle$ for $V_A \leq N \leq V/2$.

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Results ($V_A \leq N \leq V/2$)

$$\langle S_A \rangle = 1 - \frac{V_A}{V}(1+V) + V\Psi(V) - \frac{V_A}{V}[(V-N)\Psi(V-N) + N\Psi(N)] + (V_A - V)\Psi(V - V_A + 1)$$

$$=$$

Bianchi, Hackl, MK, Rigol, Vidmar (21')

Results ($V_A \leq N \leq V/2$)

$$\begin{aligned}\langle S_A \rangle =& 1 - \frac{V_A}{V}(1+V) + V\Psi(V) - \frac{V_A}{V}[(V-N)\Psi(V-N) + N\Psi(N)] \\ &+ (V_A - V)\Psi(V - V_A + 1) \\ =& V\Big((f-1)\ln(1-f) + f[(n-1)\ln(1-n) - n\ln(n) - 1]\Big) \\ &+ \frac{f[1-f+n(1-n)]}{12(1-f)(1-n)n}\frac{1}{V} + \mathcal{O}(V^{-2})\end{aligned}$$

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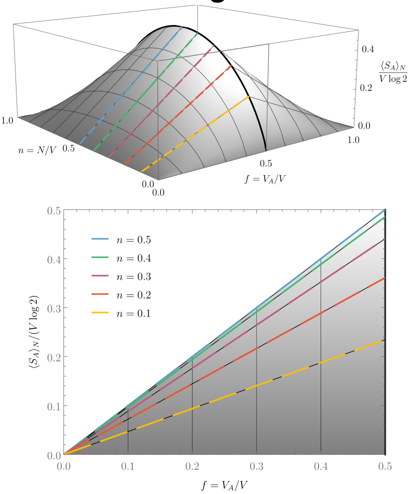
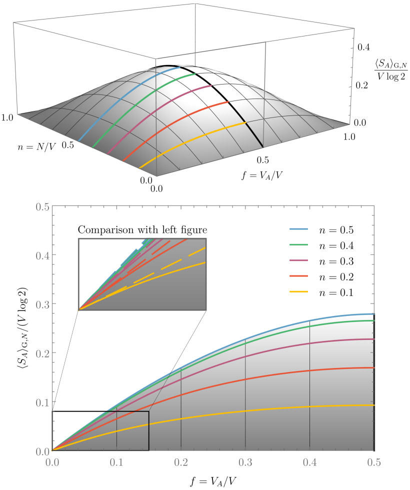
Results for fixed N ($V_A \leq N \leq V/2$)

Page



Gauss



Bianchi, Hackl, MK, Rigol, Vidmar (21')

Conclusions

- Computation of the mean $\langle S_A \rangle$ (in this talk) and standard deviation ΔS_A for general pure states (Page setting) at variable (in this talk) and fixed number of fermions N and bosons up to order $\mathcal{O}(1)$.

- Computation of the mean $\langle S_A \rangle$ (in this talk) and standard deviation ΔS_A for fermionic Gaussian pure states (Page setting) at variable and fixed number of fermions N up to order $\mathcal{O}(1/V)$.

- We have also computed the average over N with a binomial weight $\binom{V}{N} e^{-wN}$. (w is not the chemical potential though one can give it a similar interpretation.)

- Numerical simulations of spin-chains corroborate the universality of our results (even for the sub-leading orders!).

Open Questions

- Translation invariant systems show deviations from our results!
- Rigorous proofs for SYK- or many-body Hamiltonians that they follow our universal results! (Numerical evidence shows this!)
- What is the impact of the symmetry classes of the corresponding Hamiltonian? (already under investigation)
- What is with dynamical Hamiltonian (dynamical thermalisation)?

Many Thanks for your attention!

1. E. Bianchi, L. Hackl, MK (2021): arXiv:2103.05416

2. E. Bianchi, L. Hackl, MK, M. Rigol, L. Vidmar (2021): arXiv:2112.06959