Entanglement Entropy of Pure Quantum States in Fermionic Many-Body Systems

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Entanglement Entropy

System coupled to a thermal bath (equilibrium ensemble):

\[ \rho = \frac{e^{-\hat{H}/T}}{Z} = \sum_i \frac{e^{-E_i/T}}{Z} |E_i\rangle\langle E_i| \]

\[ S = -\sum_i p_i \ln p_i \]

(Thermal entropy)

images are by Lucas Hackl
Entanglement Entropy

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(Thermal entropy)

- Subsystem A in larger system (non-equilibrium situation):

\[ \rho_A = \sum_i P_i |\psi^A_i\rangle \langle \psi^A_i| = \text{Tr}_B |\psi\rangle \langle \psi| \]

\[ S_A = -\sum_i P_i \ln P_i = -\text{Tr} \rho_A \ln \rho_A \]

(Entanglement entropy)
Integrable vs Chaotic

- Integrable systems like regular quantum billiards have regular wave functions.

⇒ Entanglement entropy cannot be maximal.

⇒ Thermalisation cannot happen for integrable systems.
Integrable vs Chaotic

- Chaotic systems like chaotic quantum billiards have irregular wave functions.

⇒ Entanglement entropy can be maximal.

⇒ Thermalisation may happen for chaotic systems.

E.g., Porter-Thomas distribution for the distribution of eigenvector coefficients applies. Eigenvectors look like Haar distributed unit vectors!
Page’s Idea (based on Lubkin 78’, Pagels, Lloyd 88’)

- Eigenvectors of chaotic quantum system $|\psi\rangle \in \mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ seem to be close to Haar distributed unit vectors.

- Let $\dim \mathcal{H}_A = d_A$ and $\dim \mathcal{H}_B = d_B$ implying $\dim \mathcal{H} = d = d_A d_B$.

- Choose a Haar distributed unit vector $|\psi\rangle \in S^{2d-1} = U(d)/U(d-1)$.

- Page’s idea (93’): Consider the quantum state $\rho = |\psi\rangle \langle \psi|$ and the reduced density matrix of the subsystem $A$ is $\rho_A = \text{Tr}_B \rho$. 
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- Page’s idea (93’): Consider the quantum state \( \rho = |\psi\rangle \langle \psi| \) and the reduced density matrix of the subsystem A is \( \rho_A = \text{Tr}_B \rho \).

Conjecture:
The average entanglement entropy \( \langle S_A \rangle = \langle \text{Tr} \rho_A \ln \rho_A \rangle \) is the generic one for an eigenstate of a complex quantum system when \( d_A, d_B \to \infty \).
Pure States

- $\psi \in S^{2d-1} \in \mathcal{H}_A \otimes \mathcal{H}_B$ can be expanded:

$$|\psi\rangle = \sum_{a=1}^{d_A} \sum_{b=1}^{d_B} W_{ab} |a\rangle \otimes |b\rangle,$$

where $\{|a\rangle\} \subset \mathcal{H}_A$ and $\{|b\rangle\} \subset \mathcal{H}_B$ are orthonormal bases.

- Collect coefficients in terms of a matrix $W = \{W_{ab}\} \in \mathbb{C}^{d_A \times d_B}$.

- **Normalisation:** $||\psi|| = 1$ reads as follows

$$1 = \langle\psi|\psi\rangle = \text{Tr} W^\dagger W.$$

- **Reduced density matrix** is given as

$$\rho_A = \text{Tr}_B |\psi\rangle \langle\psi| = WW^\dagger.$$
Haar distributed Pure States

Let $|\psi\rangle \in S^{2d-1} \in \mathcal{H}_A \otimes \mathcal{H}_B$ be Haar distributed (uniformly distributed):

$$|\psi\rangle = \sum_{a=1}^{d_A} \sum_{b=1}^{d_B} W_{ab} |a\rangle \otimes |b\rangle.$$ 

Random Matrix Ensemble: $W$ is distributed by

$$P(W) = \frac{\delta(1 - \text{Tr} WW^\dagger)}{\int \delta(1 - \text{Tr} WW^\dagger) d[W]}.$$ 

This is a fixed trace ensemble!

Idea: Tracing this ensemble back to the complex Wishart-Laguerre ensemble.
Historical Results

Average entanglement entropy:

\[
\langle S_A \rangle = \psi(d_A d_B + 1) - \psi(d_B + 1) - \frac{d_A - 1}{2 d_B} = \ln(d_A) - \frac{d_A}{2 d_B} + o(1)
\]

Digamma function: \( \psi(x) = \partial \ln \Gamma(x) \)

For a qubit system:

1. \( d_A = 2^{V_A}, \ d_B = 2^{V_B} \)

2. \( V = V_A + V_B \rightarrow \infty \)

3. \( \lim_{V \rightarrow \infty} V_A / V = f \in (0, 1/2] \)

\[
\langle S_A \rangle \rightarrow \ln(2) f V - 2^{-(1-2f) V - 1}
\]

Second term is only present when \( 1 - 2f \propto 1/V \).

Page (93’), Foong, Kanno (94’); Sánchez-Rui (95’); Sen (96’)
Pure Fermionic Gaussian States
What is a Fermionic Gaussian State?

**Definition:** A fermionic Gaussian state is

\[
\rho = \exp[-\gamma Q \gamma^\dagger], \text{ with } Q = -Q^T \in i\mathbb{R}^{2V \times 2V}
\]

and \( \gamma = (\gamma_1, \ldots, \gamma_{2V}) \) are Majorana fermions meaning they build a Clifford algebra in the irreducible matrix representation in \( \mathbb{C}^{2^V \times 2^V} \)

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\gamma_a \gamma_b + \gamma_b \gamma_a = \delta_{ab} 1_{2^V}.
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Let \( \lambda_j \geq 0 \) be the singular values of \( Q \) and \( \{\eta_j\} \subset \text{span}\{\gamma_a\} \) the corresponding eigenbases of the Majorana fermions. Then,

\[ \rho = \prod_{j=1}^{V} \frac{\mathbf{1}_{2V} + 2itanh(\lambda_j) \eta_{2j-1} \eta_{2j}}{2}. \]
The correlation matrix (only antisymmetric part)

\[ J_{ab} = -J_{ba} = i \text{Tr} \left[ \rho \left( \gamma_a \gamma_b - \frac{1}{2} 1_{2V} \right) \right] \]

comprises all information of a Gaussian state.

Peschel (03'); Bianchi, Hackl (20')
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Relation to \( Q \) is \( J = i \tanh(Q) \)
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- Relation to \( Q \) is \( J = i \tanh(Q) \)

- The singular values of \( J \) are \( x_1, \ldots, x_V \in [0, 1] \) and the von Neumann entropy is

\[ S = \text{Tr}(\rho \ln \rho) = \sum_{j=1}^{V} s(x_j) \quad \text{with} \]

\[ s(x) = \frac{1 + x}{2} \ln \left( \frac{1 + x}{2} \right) + \frac{1 - x}{2} \ln \left( \frac{1 - x}{2} \right) \]

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Pure Fermionic Gaussian State = all \( x_j = 1 \)

Peschel (03'); Bianchi, Hackl (20')
Group action on pure fermionic Gaussian states is given by

\[ J \rightarrow OJO^T \text{ with } O \in O(2V) \]

Due to \( J^2 = 1_n \) and \( J = -J^T = J^* \) all correlation matrices can be written as follows

\[ J = O \begin{bmatrix} 0 & 1_V \\ -1_V & 0 \end{bmatrix} O^T \]

Subgroup satisfying \( J_0 = OJ_0O^T \) is given by \( O \in U(V) \) in the real \( 2V \times 2V \) matrix representation.

\( \Rightarrow \) Manifold of all pure fermionic Gaussian state: \( O(2V)/U(V) \)

Choose a Haar distributed \( O \in O(2V) \) to create a uniformly distributed state.

Bianchi, Hackl (20')
Reduced Density Matrix

- System $A$ is given by Majorana fermions $\gamma_1, \ldots, \gamma_{2V_A}$ and the reduced density matrix is still a Gaussian state, namely an embedded Random Matrix.

- Reduced correlation matrix $J_A$ is an orthogonal rank $V_A$ projection of $J$, namely (Bianchi, Hackl (20'))

\[
J = \begin{bmatrix}
J_A & * \\
* & * & *
\end{bmatrix}
\]

- There is still the subsystem-subsystem symmetry! Can be solved with Jacobi polynomials!
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- The jpdf of the singular values of $J_A$ for $f = V_A/V \in [0, 1/2]$ is (Bianchi, Hackl, Kieburg (21'))

\[
p(x_1, \ldots, x_{VA}) \propto \prod_{a<b} (x_b^2 - x_a^2)^2 \prod_{j=1}^{V} (1 - x_j^2)^{V-2VA}
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Average Entanglement Entropy

\[
\langle S_A \rangle = \int d[x] \rho(x) \sum_{j=1}^{V} \left[ \frac{1 + x_j}{2} \ln \left( \frac{1 + x_j}{2} \right) + \frac{1 - x_j}{2} \ln \left( \frac{1 - x_j}{2} \right) \right]
\]

Bianchi, Hackl, Kieburg (21'); leading order already found by Lydžba, Rigol, Vidmar (20')
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\[
= \lim_{\epsilon \to 1} \partial_{\epsilon} \int d[x] p(x) \sum_{j=1}^{V} \left[ \left( \frac{1 + x_j}{2} \right)^\epsilon + \left( \frac{1 - x_j}{2} \right)^\epsilon \right]
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\[ = \left( V - \frac{1}{2} \right) \psi(2V) + \left( \frac{1}{2} + V_A - V \right) \psi(2V - 2V_A) \]

\[ + \left( \frac{1}{4} - V_A \right) \psi(V) - \frac{1}{4} \psi(V - V_A) - V_A \]

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Average Entanglement Entropy

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= \left( V - \frac{1}{2} \right) \psi(2V) + \left( \frac{1}{2} + V_A - V \right) \psi(2V - 2V_A) \\
+ \left( \frac{1}{4} - V_A \right) \psi(V) - \frac{1}{4} \psi(V - V_A) - V_A \\
= V \left[ (\ln(2) - 1)f + (f - 1)\ln(1 - f) \right] + \frac{f}{2} + \frac{\ln(1 - f)}{4} + O(V^{-1})
\]

Contribution from the Page curve

Bianchi, Hackl, Kieburg (21'); leading order already found by Lydžba, Rigol, Vidmar (20')
Comparison

Reduced density matrices of Gaussian pure states are usually not maximally entangled!
Pure Gaussian Fermion States with Particle Preservation

Go over to annihilation-creation operators

$$f_j = \frac{1}{\sqrt{2}}(\gamma_{2j-1} + i\gamma_{2j})$$  
and  
$$f_j^\dagger = \frac{1}{\sqrt{2}}(\gamma_{2j-1} - i\gamma_{2j})$$
Pure Gaussian Fermion States with Particle Preservation

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Respective correlation matrix is

\[ \hat{J} = i \begin{bmatrix} 
\langle \psi | f_a f_b^\dagger - f_b^\dagger f_a | \psi \rangle & \langle \psi | f_a^\dagger f_b - f_b^\dagger f_a^\dagger | \psi \rangle \\
\langle \psi | f_a f_b - f_b f_a | \psi \rangle & \langle \psi | f_a^\dagger f_b^\dagger - f_b^\dagger f_a | \psi \rangle 
\end{bmatrix} \in \mathbb{C}^{2V \times 2V} \]
Pure Gaussian Fermion States with Particle Preservation

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Number preserving pure state yields

\[ \hat{J} = i \begin{bmatrix} \langle \psi | f_a f_b^\dagger - f_b^\dagger f_a | \psi \rangle & 0 \\ 0 & \langle \psi | f_a^\dagger f_b - f_b^\dagger f_a^\dagger | \psi \rangle \end{bmatrix} = i \begin{bmatrix} F & 0 \\ 0 & -F^T \end{bmatrix} \]

with \( F \in \text{Herm} (V) \cap \text{U} (V) \Rightarrow F^2 = 1_V \)
Go over to annihilation-creation operators

\[ f_j = \frac{1}{\sqrt{2}} (\gamma_{2j-1} + i \gamma_{2j}) \quad \text{and} \quad f_j^\dagger = \frac{1}{\sqrt{2}} (\gamma_{2j-1} - i \gamma_{2j}) \]

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with \( F \in \text{Herm}(V) \cap \text{U}(V) \Rightarrow F^2 = 1_V \)

Relation to particle number: \( \text{Tr}(F) = 2N - V \)
Manifold of all pure Gaussian states with exactly $N$ fermions is given by $U(V)/[U(N) \times U(V-N)]$. Elements are given by

$$F = U\text{diag}(1_N, -1_{V-N})U^\dagger, \quad U \in U(V)$$
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$U \in \mathbb{U}(V)$

Choose Haar a distributed $U \in \mathbb{U}(V)$ to create a uniformly distributed ensemble on $\mathbb{U}(V)/[\mathbb{U}(N) \times \mathbb{U}(V - N)]$. 

Reduced correlation matrix is given by

$$\hat{J}_A = i \text{diag}(\Pi F \Pi^T, -\Pi F^T \Pi^T)$$

with $\Pi$ is the orthogonal projection to first $V_A$ rows.

$$\Pi F \Pi^T = 2 U_A U_A^\dagger - 1_{V_A}$$

with $U_A \in \mathbb{C}^{V_A \times N}$ upper left block of $U \in \mathbb{U}(V)$. This is the complex Jacobi ensemble!
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This is the complex Jacobi ensemble!
1. Particle-hole symmetry: \( N \leftrightarrow V - N \)

2. Subsystem-subsystem symmetry: \( V_A \leftrightarrow V - V_A \)

3. Particle-subsystem symmetry: \( V_A \leftrightarrow N \leftrightarrow V_A \leftrightarrow N^\dagger \leftrightarrow U_A \leftrightarrow U_A^\dagger \)

\[
\begin{array}{c}
0 \\
\vdots \\
N \\
V_A \\
V
\end{array}
\]

graphic courtesy by Lucas Hackl

Peschel (03\'); Hackl, Bianchi (20\')
1. **Particle-hole symmetry:** $N \leftrightarrow V - N$

2. **Subsystem-subsystem symmetry:** $V_A \leftrightarrow V - V_A$

Graphic courtesy by Lucas Hackl

Peschel (03'); Hackl, Bianchi (20')
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   \( U_A U_A^\dagger \leftrightarrow U_A^\dagger U_A \)

Graphic courtesy by Lucas Hackl

Peschel (03'); Hackl, Bianchi (20')
Symmetries

1. **Particle-hole symmetry**: \( N \leftrightarrow V - N \)

2. **Subsystem-subsystem symmetry**: \( V_A \leftrightarrow V - V_A \)

3. **Particle-subsystem symmetry**: \( V_A \leftrightarrow N \)
   \( U_A U_A^\dagger \leftrightarrow U_A^\dagger U_A \)

⇒ It suffices to compute \( \langle S_A \rangle \) for \( V_A \leq N \leq V/2 \).

Peschel (03'); Hackl, Bianchi (20')

graphic courtesy by Lucas Hackl
Results \((V_A \leq N \leq V/2)\)

\[
\langle S_A \rangle = 1 - \frac{V_A}{V}(1 + V) + V\psi(V) - \frac{V_A}{V}[(V - N)\psi(V - N) + N\psi(N)] \\
+ (V_A - V)\psi(V - V_A + 1)
\]
Results \((V_A \leq N \leq V/2)\)

\[
\langle S_A \rangle = 1 - \frac{V_A}{V} (1 + V) + V \psi(V) - \frac{V_A}{V} [(V - N) \psi(V - N) + N \psi(N)]
\]

\[
+ (V_A - V) \psi(V - V_A + 1)
\]

\[
= V \left( (f - 1) \ln(1 - f) + f [ (n - 1) \ln(1 - n) - n \ln(n) - 1] \right)
\]

\[
+ \frac{f[1 - f + n(1 - n)]}{12(1 - f)(1 - n)n} \frac{1}{V} + O(V^{-2})
\]

Bianchi, Hackl, MK, Rigol, Vidmar (21')
Results for fixed $N$ ($V_A \leq N \leq V/2$)

Page

Gauss

Bianchi, Hackl, MK, Rigol, Vidmar (21')
Conclusions

▶ Computation of the mean $\langle S_A \rangle$ (in this talk) and standard deviation $\Delta S_A$ for general pure states (Page setting) at variable (in this talk) and fixed number of fermions $N$ and bosons up to order $O(1)$.

▶ Computation of the mean $\langle S_A \rangle$ (in this talk) and standard deviation $\Delta S_A$ for fermionic Gaussian pure states (Page setting) at variable and fixed number of fermions $N$ up to order $O(1/V)$.

▶ We have also computed the average over $N$ with a binomial weight $\binom{V}{N} e^{-wN}$. ($w$ is not the chemical potential though one can give it a similar interpretation.)

▶ Numerical simulations of spin-chains corroborate the universality of our results (even for the sub-leading orders!).
Open Questions

➢ Translation invariant systems show deviations from our results!

➢ Rigorous proofs for SYK- or many-body Hamiltonians that they follow our universal results! (Numerical evidence shows this!)

➢ What is the impact of the symmetry classes of the corresponding Hamiltonian? (already under investigation)

➢ What is with dynamical Hamiltonian (dynamical thermalisation)?
Many Thanks for your attention!
