The tensor Harish-Chandra–Itzykson–Zuber integral

Răzvan Gurău (CIRM, 2022)

(with B. Collins and L. Lionni)

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STRUCTURES
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1. Multipartite quantum systems

2. Asymptotic expansion of the tensor HCIZ integral

3. Scaling Ansätze and detection of entanglement

Entanglement in quantum systems

Quantum systems:

- the states are density matrices B in some Hilbert space $\mathcal{H}$
- the observables are Hermitian operators A on the Hilbert space
- the expectations are $\langle A \rangle_B = \mathrm{Tr}(AB)$

- *pure* states $B = |\Psi\rangle\langle\Psi|$
- *mixed* states $B = \sum p_i \ |\Psi_i\rangle\langle\Psi_i|$ with $p_i \geq 0, \sum p_i = 1$

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System made of D subsystems $\mathcal{H} = \bigotimes_{c=1}^{D} \overbrace{\mathcal{H}_c}^{=\mathbb{C}^N}$:

- *separable* states $B = \sum_k p_k \; B_1^{(k)} \otimes \cdots \otimes B_D^{(k)}$, with $p_k \geq 0, \sum p_k = 1$
- *entangled* states $\rightarrow$ not *separable*

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It is NP hard to decide if a state is entangled or separable.

Pure / mixed states vs. separable / entangled states

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Pure states

- can be separable $|\Psi\rangle = |\Psi_1\rangle \otimes \cdots \otimes |\Psi_D\rangle$
- *or not* $|\Psi\rangle = \sum d_{j_1 \ldots j_D} |\Psi_{j_1}\rangle \otimes \cdots \otimes |\Psi_{j_D}\rangle$

For $D = 2$ a pure state is separable iff its partial trace is pure $\Leftrightarrow$ $dd^\dagger$ has exactly one non zero eigenvalue.

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Mixed states

- can be separable $|\Psi_i\rangle = |\Psi_{i;1}\rangle \otimes \cdots \otimes |\Psi_{i;D}\rangle$
- *or not* $|\Psi_i\rangle = \sum d_{j_1 \ldots j_D} |\Psi_{i;j_1}\rangle \otimes \cdots \otimes |\Psi_{i;j_D}\rangle$

Local unitary equivalence

Equivalent states on $\mathcal{H} = \bigotimes_{c=1}^{D} \mathcal{H}_c$ obtained from B by a local change of basis:

$$\left\{ B_U = UBU^\dagger \middle| U = \bigotimes_{c=1}^{D} U_c \right\} , \qquad U_c \text{ unitary operator on } \mathcal{H}_c$$

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Entanglement – average the expectations $\mathrm{Tr}(AB)$ over B_U with tensor product Haar measure dU (or random quantum measurement):

$$\int dU \, \mathrm{Tr}(AUBU^\dagger) \, , \quad \int dU \, \left[\mathrm{Tr}(AUBU^\dagger)\right]^n$$

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Can we say anything about the state B assuming we know A and the moments?

The tensor HCIZ integral

Generating function of connected moments

$$e^{C(t,A,B)} = \int dU \; e^{t \,\mathrm{Tr}(AUBU^\dagger)} \qquad C(t,A,B) = \sum_{n\geq 1} \frac{t^n}{n!} \underbrace{C_n(A,B)}_{\text{cumulants, connected moments}}$$

Compute the cumulants $C_n(A, B)$.

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Difficult! So...

Compute the cumulants $C_n(A,B)$ at large $N = \dim(\mathcal{H}_c)$

Trace invariants

- $|i_c\rangle$ basis in $\mathcal{H}_c$ and $|i_1 \dots i_D\rangle$ tensor product basis in $\mathcal{H} = \bigotimes_{c=1}^{D} \mathcal{H}_c$
- $\langle j_1 \dots j_D \,|\, A \,|\, i_1 \dots i_D\rangle \equiv A_{j_1 \dots j_D, i_1 \dots i_D}$ matrix elements of A

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Invariant associated to D permutations over n elements $\boldsymbol{\sigma} = (\sigma_1 \dots \sigma_D)$

$$\mathrm{Tr}_{\boldsymbol{\sigma}}(A) = \sum_{\text{all indices}} \left(\prod_{s=1}^{n} A_{j_1^s \dots j_D^s, i_1^s \dots i_D^s} \right) \left(\prod_{c=1}^{D} \prod_{s=1}^{n} \delta_{i_c^s, j_c^{\sigma(s)}} \right)$$

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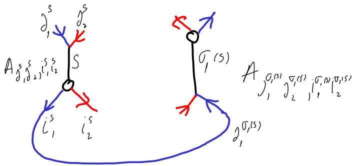

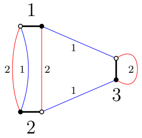

$\boldsymbol{\sigma} = \big((123), (12)(3) \big)$

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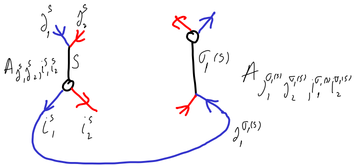

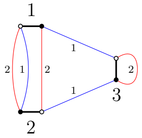

$\boldsymbol{\sigma} = \big((123), (12)(3)\big)$

If all the σ_c are the same cycle then $\mathrm{Tr}_{\boldsymbol{\sigma}}(A) = \mathrm{Tr}(A^n)$

Asymptotic expansion

Theorem (B. Collins, R.G., L. Lionni)

$$C_n(A,B) = \sum_{\boldsymbol{\sigma},\boldsymbol{\tau}} N^{-2nD+s(\boldsymbol{\sigma},\boldsymbol{\tau})}\, f[\boldsymbol{\sigma},\boldsymbol{\tau}]\, \mathrm{Tr}_{\boldsymbol{\sigma}}(A)\mathrm{Tr}_{\boldsymbol{\tau}}(B)\, \left(1+O(N^{-2})\right) ,$$

- with $s(\boldsymbol{\sigma},\boldsymbol{\tau})$ known

  $s(\boldsymbol{\sigma},\boldsymbol{\tau}) = \sum_c |\Pi(\sigma_c\tau_c)| - 2|\Pi(\boldsymbol{\sigma},\boldsymbol{\tau})| + 2$ with $|\Pi(\star)|$ the number of transitivity classes of the group generated by $\star$

- and $f[\boldsymbol{\sigma},\boldsymbol{\tau}]$ a known N independent function.

  Denoting $\nu_c = \sigma_c\tau_c$ and $\nu_{c|B}$ the restriction of ν_c to the block B of a partition $\pi_c \geq \Pi(\nu_c)$ we have:

$$f[\boldsymbol{\sigma},\boldsymbol{\tau}] = \sum_{\substack{\pi_1\geq\Pi(\nu_1),\,\dots,\,\pi_D\geq\Pi(\nu_D)\\ |\Pi(\boldsymbol{\sigma},\boldsymbol{\tau})\vee\pi_1\vee\dots\vee\pi_D|=1\\ \sum_c \Pi(\nu_c)-\sum_c|\pi_c|-|\Pi(\boldsymbol{\sigma},\boldsymbol{\tau})|+1=0}} (-1)^{nD-\sum_{c=1}^D |\Pi(\nu_c)|}$$

$$\times \prod_{p=1}^{n}\left(\frac{(2p)!}{p!(p-1)!}\right)^{\sum_c \overbrace{d_p(\nu_c)}^{\text{no. cycles of length } p}} \prod_{c=1}^{D}\prod_{B\in\pi_c}\frac{(2|B|+|\Pi(\nu_{c|B})|-3)!}{(2|B|)!} .$$

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If all σ_c are the same cycle and $\tau_c = \sigma_c^{-1}$ then

- $s[\boldsymbol{\sigma},\boldsymbol{\tau}] = Dn$ as $\nu_c = \sigma_c\tau_c$ is the identity
- $f[\boldsymbol{\sigma},\boldsymbol{\tau}] = 1$ as only π_c the partition in one element sets contribute

Large N regimes

Assume asymptotic scaling $\lim_{N\to\infty} \; N^{-s_B(\boldsymbol{\tau})}\mathrm{Tr}_{\boldsymbol{\tau}}(B) = \mathrm{tr}_{\boldsymbol{\tau}}(b) < \infty$

$$C_n(A, B) \approx \sum_{\boldsymbol{\sigma},\boldsymbol{\tau}} N^{-2nD+s(\boldsymbol{\sigma},\boldsymbol{\tau})+s_A(\boldsymbol{\sigma})+s_B(\boldsymbol{\tau})} \, f[\boldsymbol{\sigma},\boldsymbol{\tau}] \, \mathrm{tr}_{\boldsymbol{\sigma}}(a)\mathrm{tr}_{\boldsymbol{\tau}}(b)$$

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Large N regime $-$ constants δ and γ such that:

$$\forall n \qquad \lim_{N\to\infty} \frac{1}{N^{\delta}} \, N^{n\gamma} \; C_n(A,B) = c_n(a,b) = \sum_{\substack{\boldsymbol{\sigma},\boldsymbol{\tau} \\ h(\boldsymbol{\sigma},\boldsymbol{\tau})=0}} f[\boldsymbol{\sigma},\boldsymbol{\tau}] \; \mathrm{tr}_{\boldsymbol{\sigma}}(a)\mathrm{tr}_{\boldsymbol{\tau}}(b) \; .$$

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Distinguished by the leading order invariants $\{(\boldsymbol{\sigma},\boldsymbol{\tau}) | h(\boldsymbol{\sigma},\boldsymbol{\tau}) = 0\}$

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Distinguished by the leading order invariants $\{(\boldsymbol{\sigma},\boldsymbol{\tau}) | h(\boldsymbol{\sigma},\boldsymbol{\tau}) = 0\}$

$$\lim_{N\to\infty} \frac{1}{N^{\delta}} \ln\left(\int dU \; e^{tN^{\gamma}\mathrm{Tr}(AUBU^{\dagger})} \right) = \sum_{n\geq 1} \frac{t^n}{n!} c_n(a,b)$$

Large N scaling of invariants

Observable $\mathrm{Tr}_{\boldsymbol{\sigma}}(A) \sim N^{s_A(\boldsymbol{\sigma})}\mathrm{tr}_{\boldsymbol{\sigma}}(a)$ and state $\mathrm{Tr}_{\boldsymbol{\tau}}(B) \sim N^{s_B(\boldsymbol{\tau})}\mathrm{tr}_{\boldsymbol{\tau}}(b)$

- a local observable is a tenosr product of low rank operators $A = \bigotimes_{c=1}^{D} A_c$, that is $\mathrm{Tr}_{\boldsymbol{\sigma}}(A) \sim O(1)$ and $s_A(\boldsymbol{\sigma}) = 0$

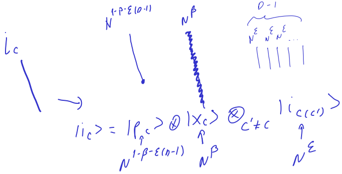
- state B has a "separated" and a "entangled" part

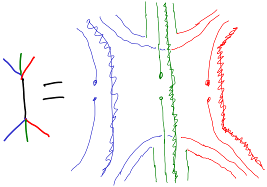
$$s_B(\boldsymbol{\tau}) = \alpha n + \beta \sum_c \text{something that sees only } c + \epsilon \sum_{c_1, c_2} \text{something that mixes } c_1, c_2 + \epsilon' \sum_{c_1,c_2,c_3} \text{something that mixes } c_1, c_2, c_3 + \dots$$

A family of states

Split $\mathcal{H}_c = \mathcal{H}_c^1 \otimes \mathcal{H}_c^s \otimes \mathcal{H}_c^e$ such that $\mathcal{H}_c^e$ splits further into $D-1$ factors $\mathcal{H}_c^e = \bigotimes_{c' \neq c} \mathcal{H}_{c(c')}$:

$$|i_c\rangle = \underbrace{|p_c\rangle}_{\dim(\mathcal{H}_c^1)=N^{1-\beta-\epsilon(D-1)}} \otimes \underbrace{|X_c\rangle}_{\dim(\mathcal{H}_c^s)=N^{\beta}} \bigotimes_{c'\neq c} \underbrace{|i_{c(c')}\rangle}_{\dim(\mathcal{H}_{c(c')})=N^{\epsilon}}$$





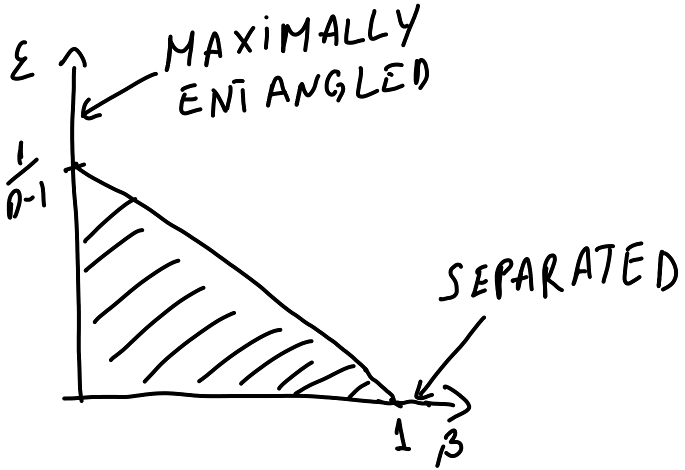
Choose $B = N^{-\beta D - \epsilon \binom{D}{2}} B^1 \otimes B^s \otimes B^e$ with:

$$B^1 = \bigotimes_c \underbrace{|\Psi_c\rangle\langle\Psi_c|}_{\text{1 dim. proj. on } \mathcal{H}_c^1} \quad , \qquad B^S = \bigotimes_c \underbrace{\sum_{X_c} |X_c\rangle\langle X_c|}_{\text{indentity on } \mathcal{H}_c^s}$$

$$B^e = \underbrace{|\Psi^e\rangle\langle\Psi^e|}_{\text{pure entangled state}} \quad , \qquad |\Psi^e\rangle = \left(\prod_{c_1<c_2} \delta_{i_{c_1(c_2)} i_{c_2(c_1)}}\right) \bigotimes_{\substack{c,c' \\ c\neq c'}} |i_{c(c')}\rangle$$

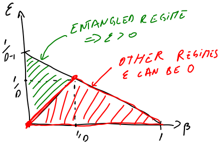
The (β, ϵ) plane

$$\mathrm{Tr}_{\boldsymbol{\tau}}(B) = N^{-n\beta D - n\epsilon\binom{D}{2} + \beta\sum_c |\Pi(\tau_c)| + \epsilon\sum_{c_1<c_2}|\Pi(\tau_{c_1}\tau_{c_2}^{-1})|}$$



Asymptotic regimes

$$\mathrm{Tr}_{\boldsymbol{\tau}}(B) \sim N^{\beta \sum_c (|\Pi(\tau_c)| - n) + \epsilon \sum_{c_1 < c_2} (|\Pi(\tau_{c_1} \tau_{c_2}^{-1})| - n)} \, \mathrm{tr}_{\boldsymbol{\tau}}(b) \,, \qquad \mathrm{Tr}_{\boldsymbol{\sigma}}(A) \sim O(1)$$



In the "entangled regime" at leading order all the σ_c are the same cycle and $\tau_c = \sigma_c^{-1}$:

$$\lim_{N \to \infty} N^{nD} C_n(A, B) = (n-1)! \, \mathrm{Tr}(A^n) \mathrm{Tr}(B^n)$$

If B respects this asymptotic scaling ansatz, A is a local observable *and* the cumulants behave like

$$C_n(A, B) \approx \frac{(n-1)!}{N^{nD}} \mathrm{Tr}(A^n) \mathrm{Tr}(B^n)$$

then $\epsilon > 0$ hence the state B is entangled.

Conclusions

Entanglement detection criterion

If for a local observable A the cumulants under averaging over local unitary transformations behave like:

$$C_n(A, B) \approx \frac{(n-1)!}{N^{nD}} \mathrm{Tr}(A^n)\mathrm{Tr}(B^n) \ ,$$

then the state B is entangled.

- the entangled regime is the *simplest* one with the fewest leading order contributions
- generalize to other asymptotic entangled scaling (so far robust: whenever the strength of entanglement ϵ is large enough we fall into the trivial regime)
- we are describing the system of D q-Nits. How about many q-bits (large D, $N = 2$)?