ALGEBRAIC-GEOMETRIC CHARACTERIZATION OF TRIPARTITE ENTANGLEMENT

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ABSTRACT

To characterize entanglement of tripartite $\mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$ systems, we employ algebraic-geometric tools that are invariants under Stochastic Local Operation and Classical Communication (SLOCC), namely k-secant varieties and one-multilinear ranks (one-multiranks). Indeed, by means of them, we present a classification of tripartite pure states in terms of a finite number of families and subfamilies. At the core of it stands out a fine-structure grouping of three-qutrit entanglement.

THE FRAMEWORK OF ALGEBRAIC GEOMETRY

Secant & Tangent Varieties

The space of states $|\psi\rangle = \sum_{i\in\{0,\dots,d-1\}^3} \mathfrak{c}_i|i\rangle$ that are fully separable has the structure of a Segre variety which is embedded in the ambient space as follows

$$\Sigma^3_{\mathbf{d-1}}:\ \mathbb{P}^{d-1}\times\mathbb{P}^{d-1}\times\mathbb{P}^{d-1}\hookrightarrow\mathbb{P}^{d^3-1}\,.$$

- A k-secant of the Segre variety joins its k points, each of which represents a separable state. It corresponds to an entangled state being a superposition of k separable states.

k-secant variety $\sigma_k(\Sigma^3_{\mathbf{d-1}}) \equiv$ union of k-secants of the Segre variety

> k-secant varieties are SLOCC invariants.

The higher k-secant fill the ambient space $\mathbb{P}(\mathbb{C}^{d^{\otimes 3}})$ when $k=\lceil\frac{d^3}{3d-2}\rceil$, except for $d=3$ where the generic rank is 5.
The proper k-secant, i.e. the set $\sigma_k(\Sigma^3_{\mathbf{d-1}})\setminus\sigma_{k-1}(\Sigma^3_{\mathbf{d-1}})$, is the union of the k-secant hyperplanes $\mathcal{S}_k\subset\sigma_k(\Sigma^n_{\mathbf{1}})$ represented by

$$\mathcal{S}_k=\sum_{i=1}^{k}\lambda_i p_i\,,\quad\{\lambda_i\}_{i=1}^k\neq 0\,,\quad\{p_i\}_{i=1}^k \text{ are distinct points } \in\Sigma^3_{\mathbf{d-1}}\,.$$

- Tangents are limits of secants, e.g., when one point tends to another one.

Tensor Rank & Border Rank

- The rank of a tensor ψ is defined as the minimum number of simple tensors (fully separable states) that sum to ψ.
- The (tensor) border rank of a tensor ψ is defined as the smallest r such that ψ is a limit of tensors of rank r.

Example: $\quad |\mathrm{W}_3\rangle = |001\rangle+|010\rangle+|100\rangle = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left((|0\rangle+\varepsilon|1\rangle)^{\otimes 3}-|000\rangle\right).$

One-Multiranks

Matricization: Reshaping 3-fold tensor product space $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2\otimes\mathcal{H}_3$ $(\mathcal{H}_i\simeq\mathbb{C}^d)$ to $\mathcal{H}\simeq\mathcal{H}_I\otimes\mathcal{H}_{\bar{I}}$, where $\mathcal{H}_I=\mathbb{C}^d$, $\mathcal{H}_{\bar{I}}=\mathbb{C}^{d^2}$, and $I=(i)$ so that $I\cup\bar{I}=(1,2,3)$.
Using Dirac notation, the flattening (matricization) of $|\psi\rangle\in\mathcal{H}$ reads

$$\mathcal{M}_I[\psi]=\left(\langle e_0|\psi\rangle,\dots,\langle e_{d-1}|\psi\rangle\right)^{\mathrm{T}}\,,\qquad \text{Matrix Order}=d\times d^2\,,$$

where $|e_j\rangle=|j\rangle$ is the computational basis of $\mathcal{H}_I$ and T denotes the matrix transposition.

> One-multiranks are SLOCC invariants.

- A state is genuinely entangled iff all one-multiranks are greater than one.
- One-multiranks of a given tensor in the k-secant are at most k.

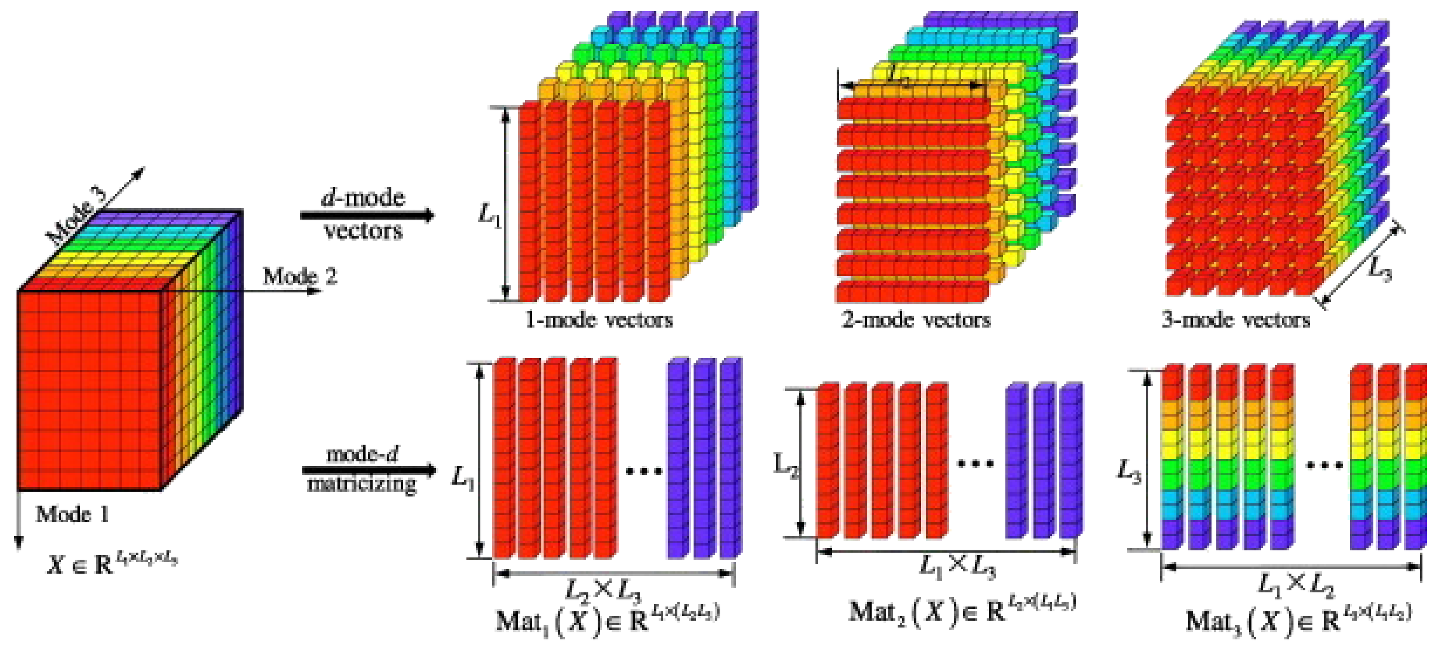

Figure 1: Flattening of a 3-order tensor to three different matrices [https://doi.org/10.1016/j.isprsjprs.2013.06.001].

Classification Algorithm

i) find families by identifying $\Sigma^3_{\mathbf{d-1}},\sigma_2(\Sigma^3_{\mathbf{d-1}}),\dots,\sigma_k(\Sigma^3_{\mathbf{d-1}})$; **ii)** split families to secants and tangents by identifying $\tau_2(\Sigma^3_{\mathbf{d-1}}),\dots,\tau_k(\Sigma^3_{\mathbf{d-1}})$; **iii)** find subfamilies by identifying one-multiranks.

EXAMPLE: 3-QUTRIT ENTANGLEMENT

In addition to the standard flattenings, for 3-qutrit systems, we have another flattening map as follows:

$$\mathcal{F}:\ \mathcal{H}_1\otimes\mathcal{H}_2^*\to\Lambda^2\mathcal{H}_1\otimes\mathcal{H}_3\,,$$

$$\mathcal{F}=\begin{pmatrix}
0 & 0 & 0 & \mathfrak{c}_0 & \mathfrak{c}_1 & \mathfrak{c}_2 & -\mathfrak{c}_9 & -\mathfrak{c}_{10} & -\mathfrak{c}_{11}\\
0 & 0 & 0 & \mathfrak{c}_3 & \mathfrak{c}_4 & \mathfrak{c}_5 & -\mathfrak{c}_{12} & -\mathfrak{c}_{13} & -\mathfrak{c}_{14}\\
0 & 0 & 0 & \mathfrak{c}_6 & \mathfrak{c}_7 & \mathfrak{c}_8 & -\mathfrak{c}_{15} & -\mathfrak{c}_{16} & -\mathfrak{c}_{17}\\
-\mathfrak{c}_0 & -\mathfrak{c}_1 & -\mathfrak{c}_2 & 0 & 0 & 0 & \mathfrak{c}_{18} & \mathfrak{c}_{19} & \mathfrak{c}_{20}\\
-\mathfrak{c}_3 & -\mathfrak{c}_4 & -\mathfrak{c}_5 & 0 & 0 & 0 & \mathfrak{c}_{21} & \mathfrak{c}_{22} & \mathfrak{c}_{23}\\
-\mathfrak{c}_6 & -\mathfrak{c}_7 & -\mathfrak{c}_8 & 0 & 0 & 0 & \mathfrak{c}_{24} & \mathfrak{c}_{25} & \mathfrak{c}_{26}\\
\mathfrak{c}_9 & \mathfrak{c}_{10} & \mathfrak{c}_{11} & -\mathfrak{c}_{18} & -\mathfrak{c}_{19} & -\mathfrak{c}_{20} & 0 & 0 & 0\\
\mathfrak{c}_{12} & \mathfrak{c}_{13} & \mathfrak{c}_{14} & -\mathfrak{c}_{21} & -\mathfrak{c}_{22} & -\mathfrak{c}_{23} & 0 & 0 & 0\\
\mathfrak{c}_{15} & \mathfrak{c}_{16} & \mathfrak{c}_{17} & -\mathfrak{c}_{24} & -\mathfrak{c}_{25} & -\mathfrak{c}_{26} & 0 & 0 & 0
\end{pmatrix}.$$

> $\lceil\frac{\mathrm{rank}\mathcal{F}}{2}\rceil$ indicate the secant family of a given state.

Table I. Fine-structure classification of 3-qutrit entanglement

$\Sigma^3_{\mathbf{2}}$	σ_2	τ_2	σ_3	τ_3	σ_4	σ_5
$\lvert\mathrm{Sep}\rangle$	$\lvert\mathrm{GHZ}_3^{(1)}\rangle$	$\lvert\mathrm{W}_3\rangle$	$\lvert\mathrm{GHZ}_3^{(2)}\rangle$	$\lvert(333)'_3\rangle$	$\lvert(333)_4\rangle$	$\lvert(333)_5\rangle$
	$\lvert\mathrm{B}_i^{(1)}\rangle_{i=1}^3$		$\lvert(332)\rangle$	$\lvert(332)'\rangle$		
			$\lvert(323)\rangle$	$\lvert(323)'\rangle$		
			$\lvert(233)\rangle$	$\lvert(233)'\rangle$		
			$\lvert(322)\rangle$			
			$\lvert(232)\rangle$			
			$\lvert(223)\rangle$			
			$\lvert\mathrm{B}_i^{(2)}\rangle_{i=1}^3$			

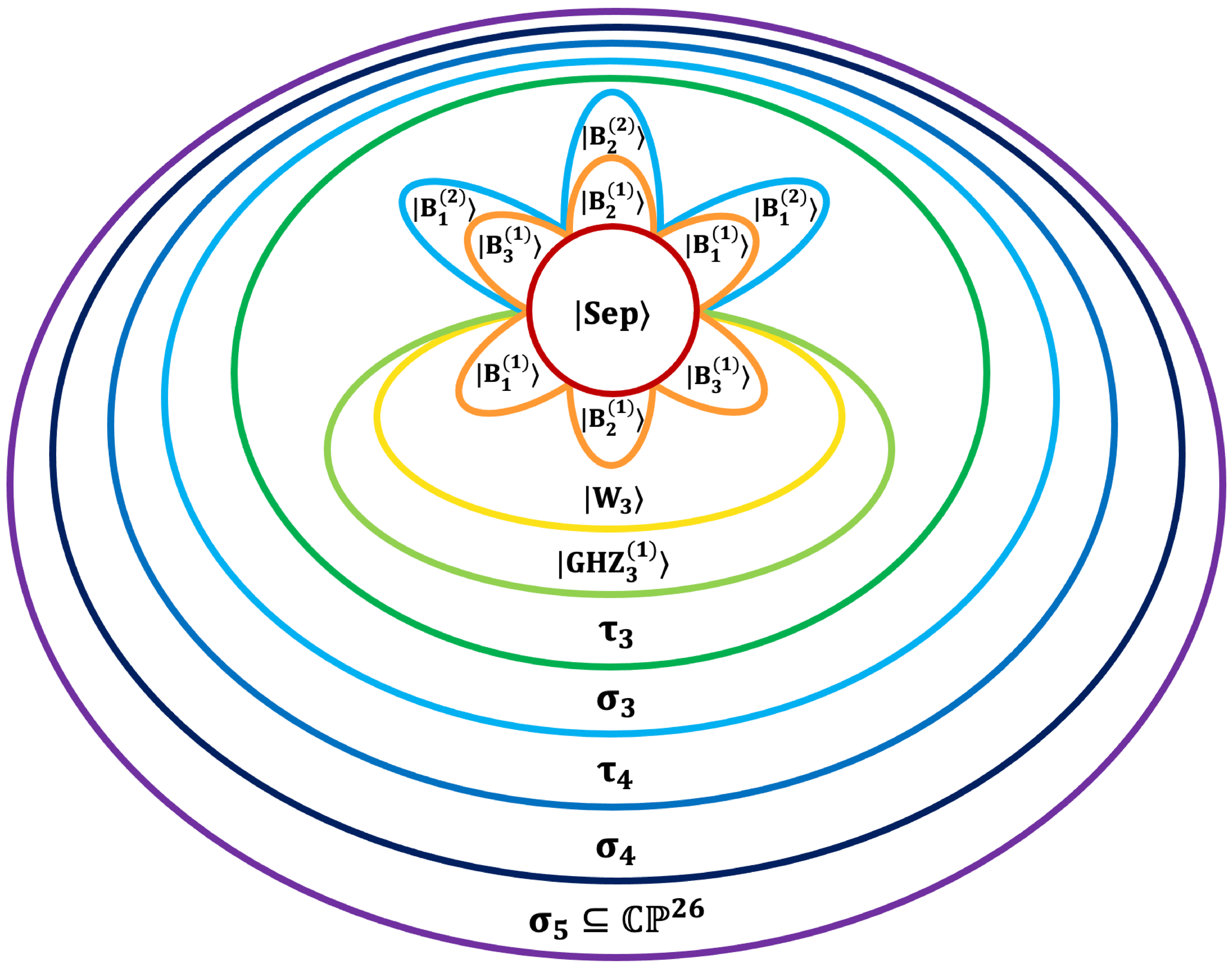

Figure 2: Petal-like classification of SLOCC orbits of 3-qutrit states. By noninvertible SLOCC one can go from the outer classes to the inner ones (from σ_k to τ_k also in an approximate way), thus generating the entanglement hierarchy. Note that states $|\mathrm{B}_i^{(1)}\rangle$ appear with a double petal because to emphasize that they can be obtained starting from either $|\mathrm{W}_3\rangle$ states or $|\mathrm{B}_i^{(2)}\rangle$ states. In contrast, $|\mathrm{B}_i^{(2)}\rangle$ states cannot be obtained from $|\mathrm{W}_3\rangle$ states.

FINER CLASSIFICATION

Finer classification of three-qutrit entanglement:

$$|\mathrm{Y}_3\rangle=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\left(\left(|0\rangle+\frac{\varepsilon}{\sqrt{2}}|1\rangle+\varepsilon^2|2\rangle\right)^{\otimes 3}+\left(|0\rangle-\frac{\varepsilon}{\sqrt{2}}|1\rangle\right)^{\otimes 3}-2|000\rangle\right),$$

$$|\mathrm{X}_3\rangle=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left((|0\rangle+\varepsilon|1\rangle)^{\otimes 3}+\varepsilon|111\rangle-|000\rangle\right).$$

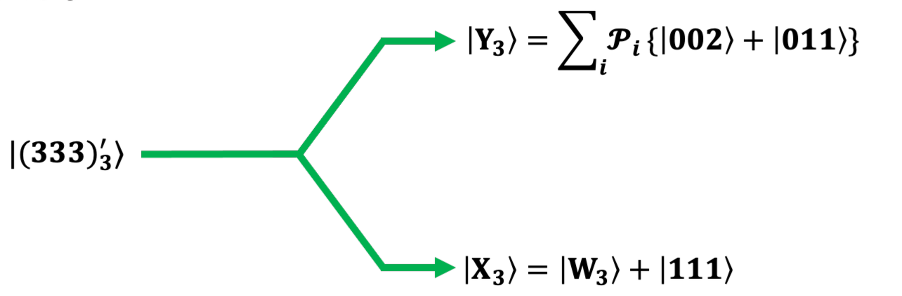

Figure 3: Using tensor rank as the third SLOCC invariant, the subfamily $|(333)'_3\rangle$ of Table I can be split into two sub-subfamilies $|\mathrm{X}_3\rangle$ and $|\mathrm{Y}_3\rangle$ with tensor ranks equal to four and five, respectively.

CONCLUSION

- One can always use n-qudit classification as a partial classification of $(n+1)$-qudit systems.
- Operational meaning (tensor rank and border rank, can be seen as the generalized Schmidt rank and its counterpart).
- This kind of classification can be considered as a reference to study (asymptotic) SLOCC interconversions among different resources based on tensor (border) rank.
- Extension of this classification to mixed states ...

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