Approximation algorithms for the geometric measure of entanglement of random multipartite quantum states

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Normalized Gradient Descent

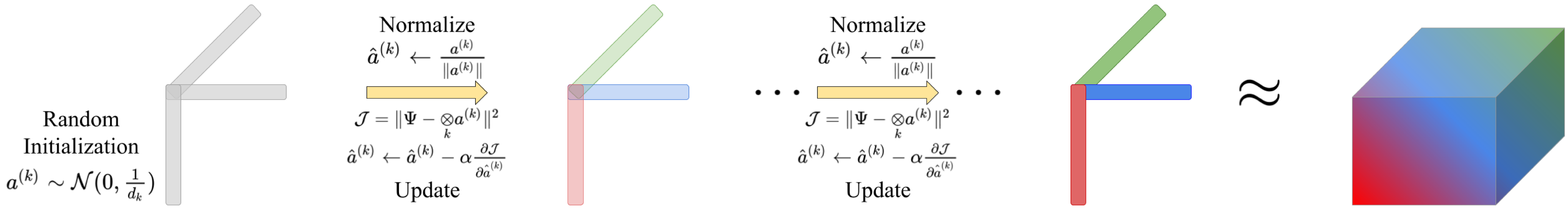


Illustration of our algorithm for a rank 1 approximation.

We propose a novel normalized gradient descent algorithm for calculating finite rank approximations of tensors. Unlike Alternating Least Squares (ALS), our algorithm incorporates normalization on each step of optimization. This mitigates the problem of sub-optimal lower bounds on the injective norm introduced due to normalization post-convergence in ALS. Our algorithm outperforms ALS with and without symmetry and normalization.

Gaussian Tensors

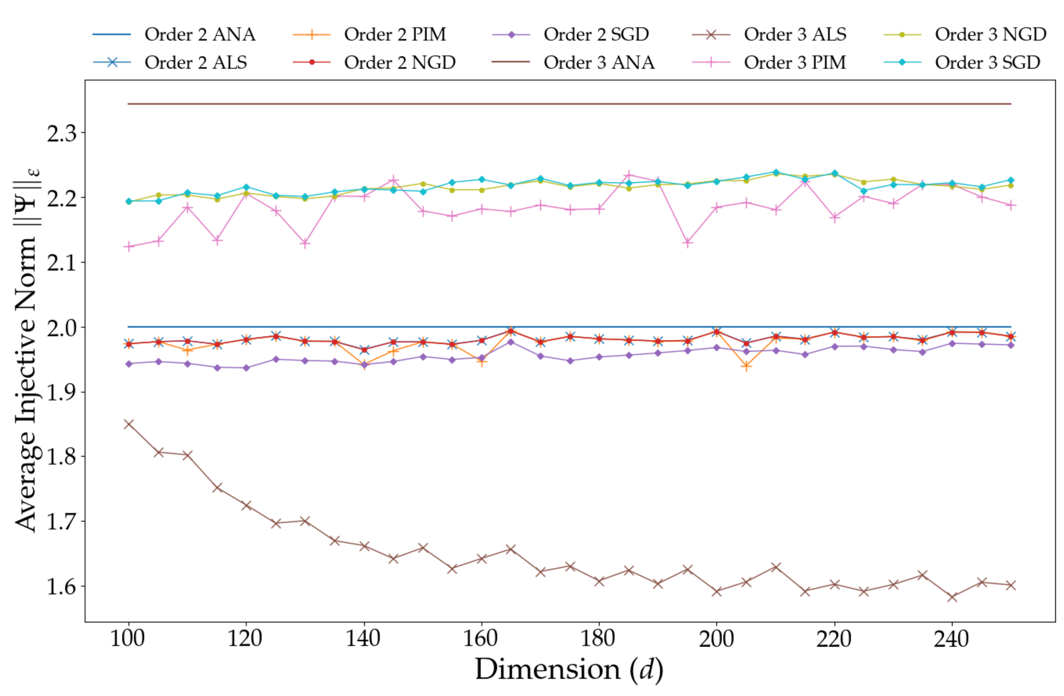
We form Gaussian tensors $X \in (\mathbb{R}^d)^{\otimes n}$ by sampling each of its entries from $\mathcal{N}(0, \frac{1}{d})$.

$$X'_{i_1,i_2,\cdots,i_n} \sim \mathcal{N}\left(0, \frac{1}{d}\right) \; \forall\, i_j \in \{1,2,\cdots,d\} \; \forall\, j \in \{1,2,\cdots,n\}$$

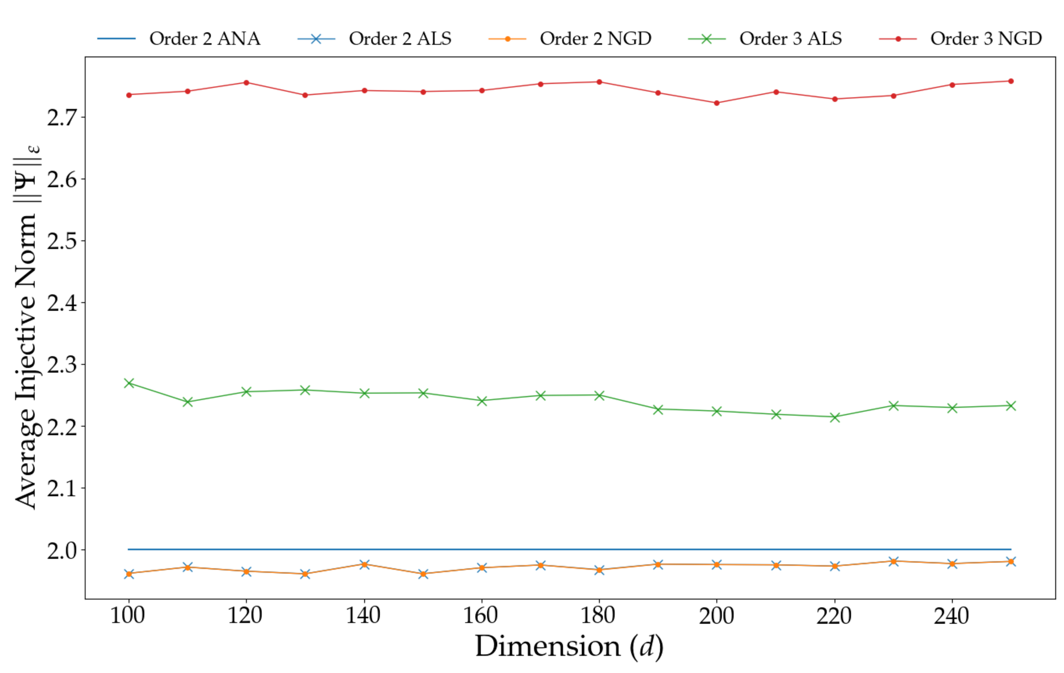
The expected value of the Euclidean norm grows as $(\sqrt{d})^{n-1}$ for large d. In the symmetric case, we incorporate a factor of 2 in the variance while sampling in accordance with the setting presented in previous works [1]. The expected value of the Euclidean norm grows as $\sqrt{\frac{2}{n!}}(\sqrt{d})^{n-1}$ for large d.

$$X_{i_1,i_2,\cdots,i_n} = \frac{1}{n!}\sum_{\sigma \in S_n} X'_{i_{\sigma(1)},i_{\sigma(2)},\cdots,i_{\sigma(n)}} \; \forall\, i_j \in \{1,2,\cdots,d\} \; \forall\, j \in \{1,2,\cdots,n\}$$

Results



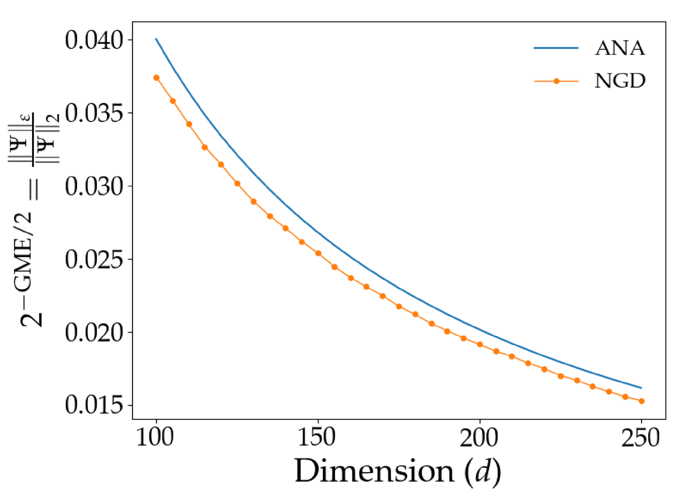
Symmetrized Gaussian tensors.



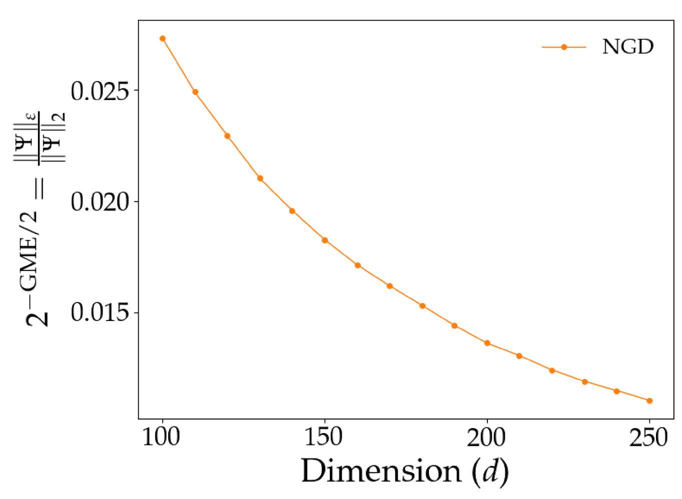
Non-symmetrized Gaussian tensors.

We benchmark all algorithms on symmetrized Gaussian tensors as the asymptotic limits of their injective norms are known [1]. Although analytical values are unknown for the non-symmetrized case, all algorithms estimate a lower bound and hence our method performs best in all the cases.

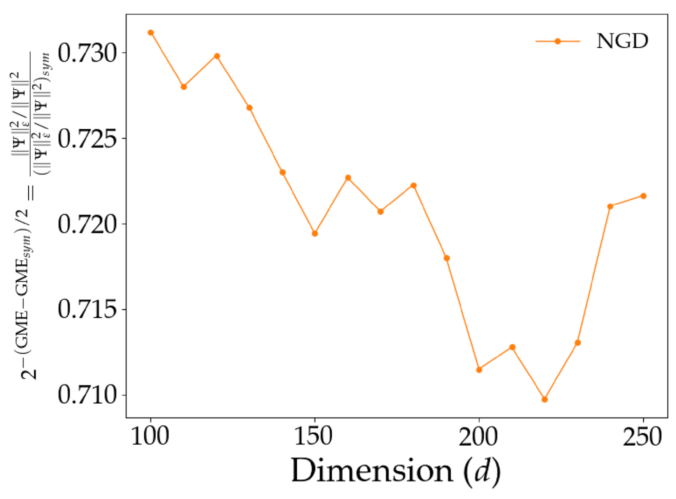
Effect of Symmetrization on Entanglement



Symmetrized Gaussian tensors.



Non-symmetrized Gaussian tensors.



Relative ratios.

The value of $2^{-\,\mathrm{GME}/2}$ for symmetrized Gaussian tensors is greater than or equal to that for non-symmetrized Gaussian tensors of the same order and dimension (we show $n = 3$). This indicates that projecting a random Gaussian tensor onto the symmetric subspace reduces entanglement.

Minimization Objective

The extent to which a quantum state is entangled can be estimated by the angle it makes with the closest product state. The *injective norm* of a tensor $\Psi \in \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}$ is defined as

$$\|\Psi\|_\varepsilon = \max_{\phi \in \mathrm{PRO}} |\langle \Psi|\phi\rangle|$$

where PRO is the set of all product states in $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \otimes \cdots \otimes \mathbb{C}^{d_n}$. The *geometric measure of entanglement* (GME) [2, 3, 4] is defined as

$$\mathrm{GME}(\Psi) = -\log_2\left(\frac{\|\Psi\|_\varepsilon^2}{\|\Psi\|^2}\right)$$

It is a faithful measure of entanglement and gives 0 only for product states. Further,

$$\begin{aligned}\min_{\phi\in\mathrm{PRO}} \|\Psi - \phi\|^2 &= \min_{\phi\in\mathrm{PRO}} (\|\Psi\|^2 + \|\phi\|^2 - 2\langle\Psi|\phi\rangle) \\ &= \min_{\phi\in\mathrm{PRO}} (\|\Psi\|^2 + 1 - 2\langle\Psi|\phi\rangle)\end{aligned}$$

Since $\|\Psi\|$ is fixed for an initial choice of Ψ, the minimization objective is equivalent to optimizing for the injective norm $\|\Psi\|_\varepsilon$

$$\min_{\phi\in\mathrm{PRO}} \|\Psi - \phi\|^2 \equiv \|\Psi\|^2 + 1 - 2\max_{\phi\in\mathrm{PRO}} |\langle\Psi|\phi\rangle|$$

Conclusions

- We propose a novel normalized gradient descent algorithm for approximating injective norms of tensors.
- Our algorithm outperforms all other algorithms for estimating the injective norms of symmetrized and non-symmetrized Gaussian tensors.
- Unlike ALS, our algorithm is robust to symmetrization and performs equally well on tensors with and without symmetry.
- We show that projecting a random Gaussian tensor onto the symmetric subspace reduces entanglement in the tensor.
- We aim to extend our results for more complicated tensor states including complex Gaussian matrix product states.

References

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