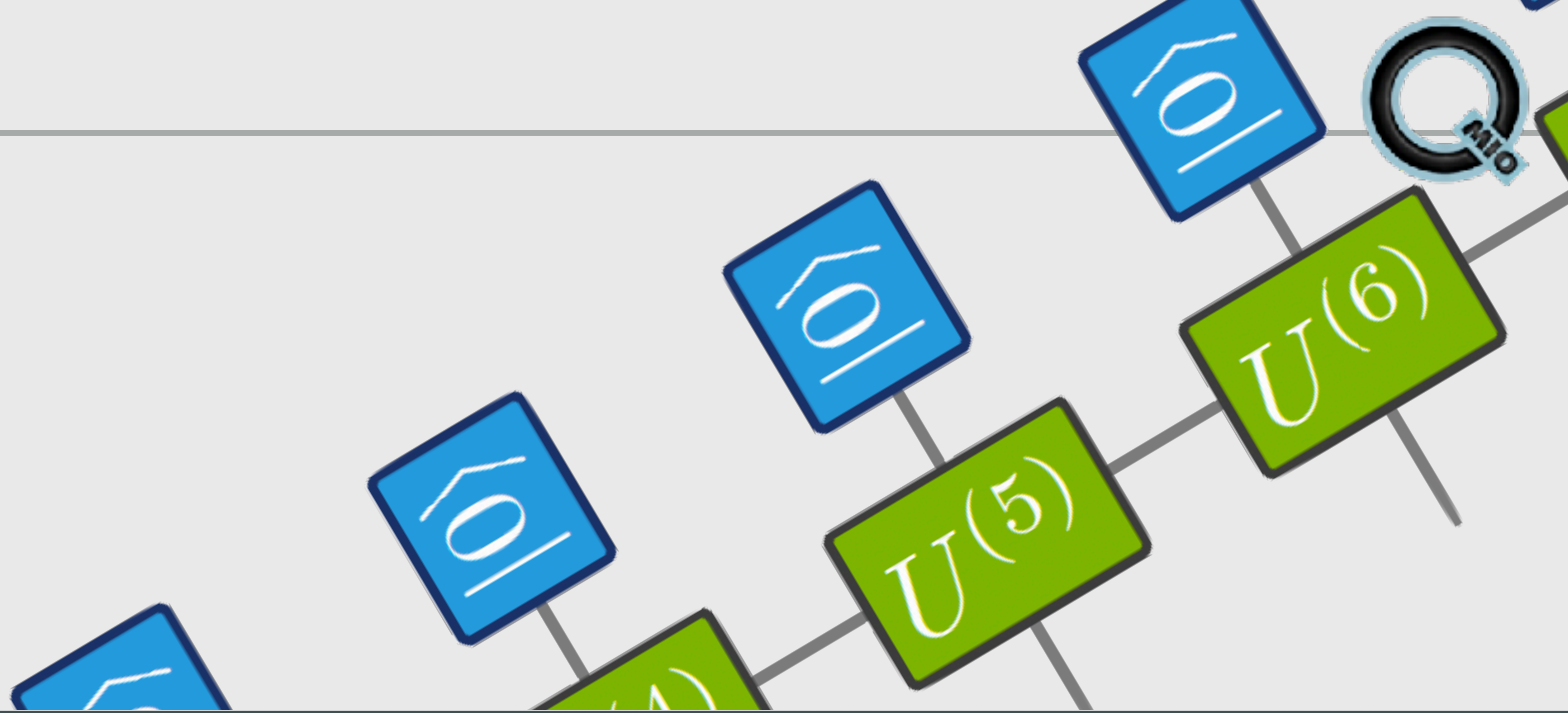


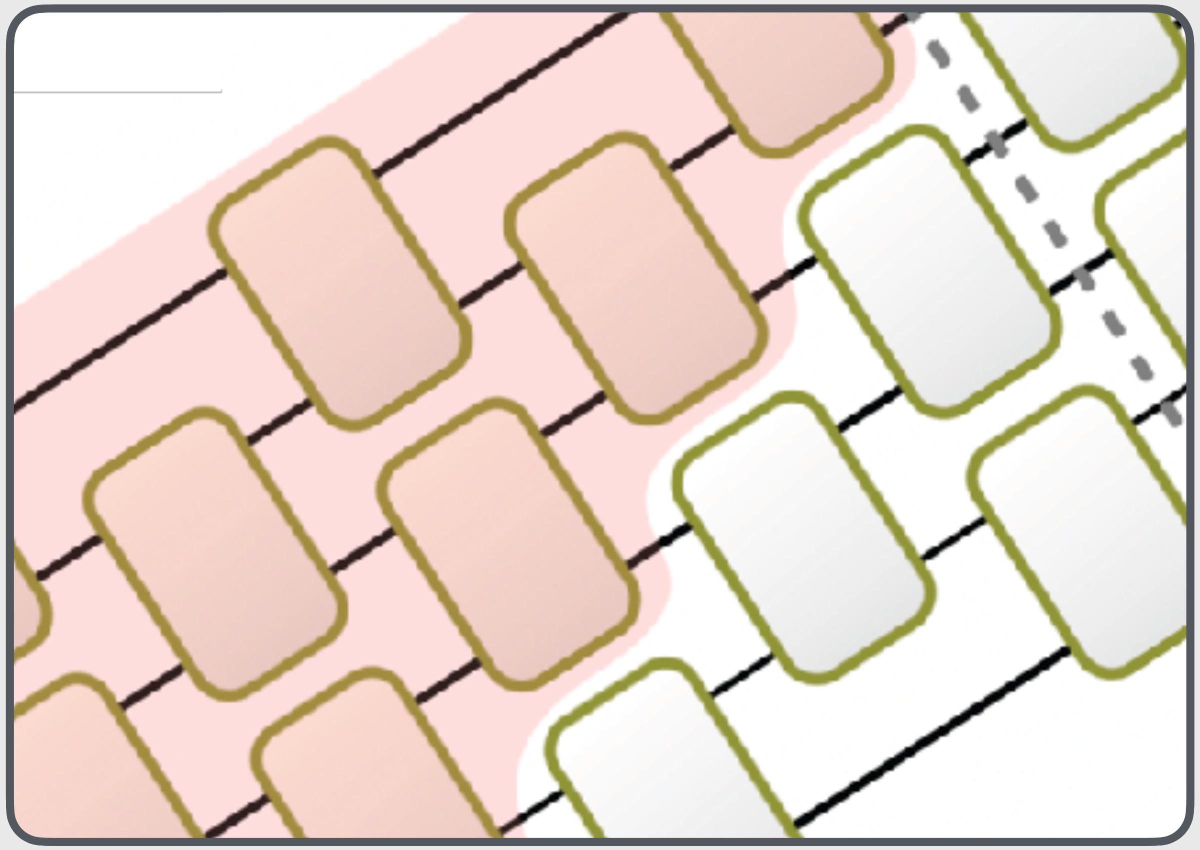
RANDOM TENSOR NETWORKS

FROM STATISTICAL MECHANICS OVER COMPLEXITY TO HOLOGRAPHY

JENS EISERT, FU BERLIN

MANIFESTO OF THIS TALK

- **Random objects** are proxies for complex statics and dynamics

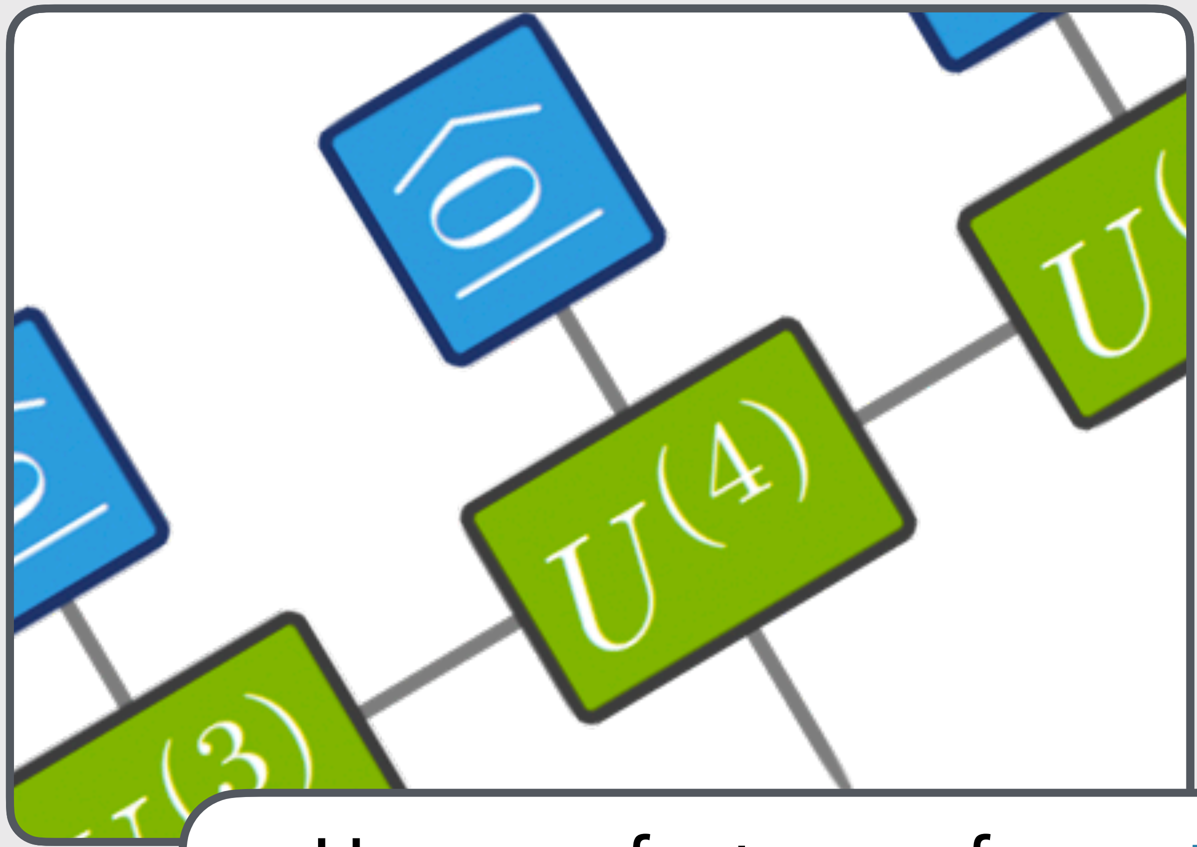


- **Random circuits** show features of *quantum chaotic dynamics* (OTOC), many-body *localization* etc
- **Random tensor networks** for *typical states in phases of matter, holographic prescriptions*
- Reminiscent of **random coding**

- Randomness as a powerful tool: Can **prove** statements fully out of reach otherwise

Brown, Fawzi, Commun Math Phys 340, 867-900 (2015)
Brandao, Harrow, Horodecki, Commun Math Phys 346, 397 (2016)
Sünderhauf, Pérez-García, Huse, Schuch, Cirac, Phys Rev B 98, 134204 (2018)
Bertini, Piroli, Phys Rev B 102, 064305 (2020)

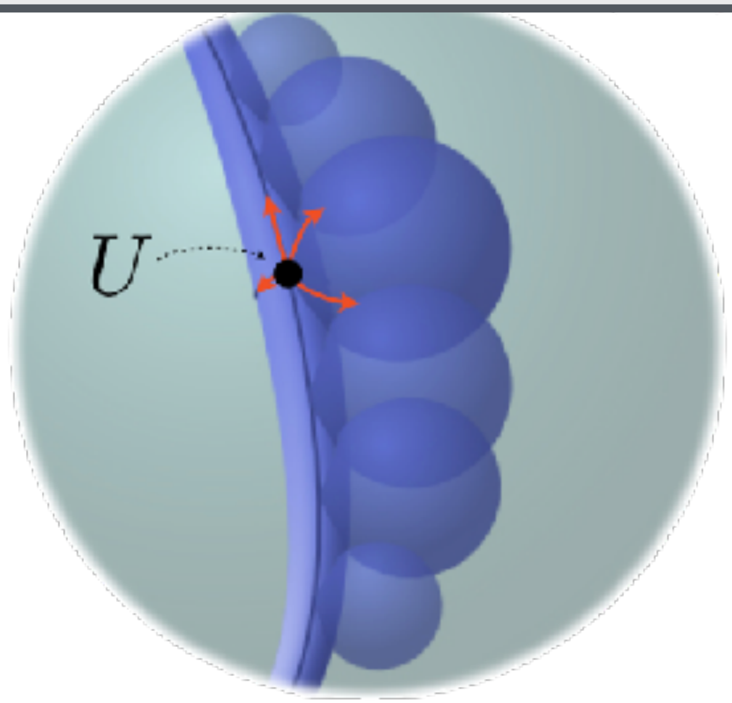
- **Random tensor networks** as typical representatives of phases of matter



- How can features of **quantum statistical mechanics** be proven?

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)

- **Brown Susskind conjecture** on linear complexity growth of random circuits



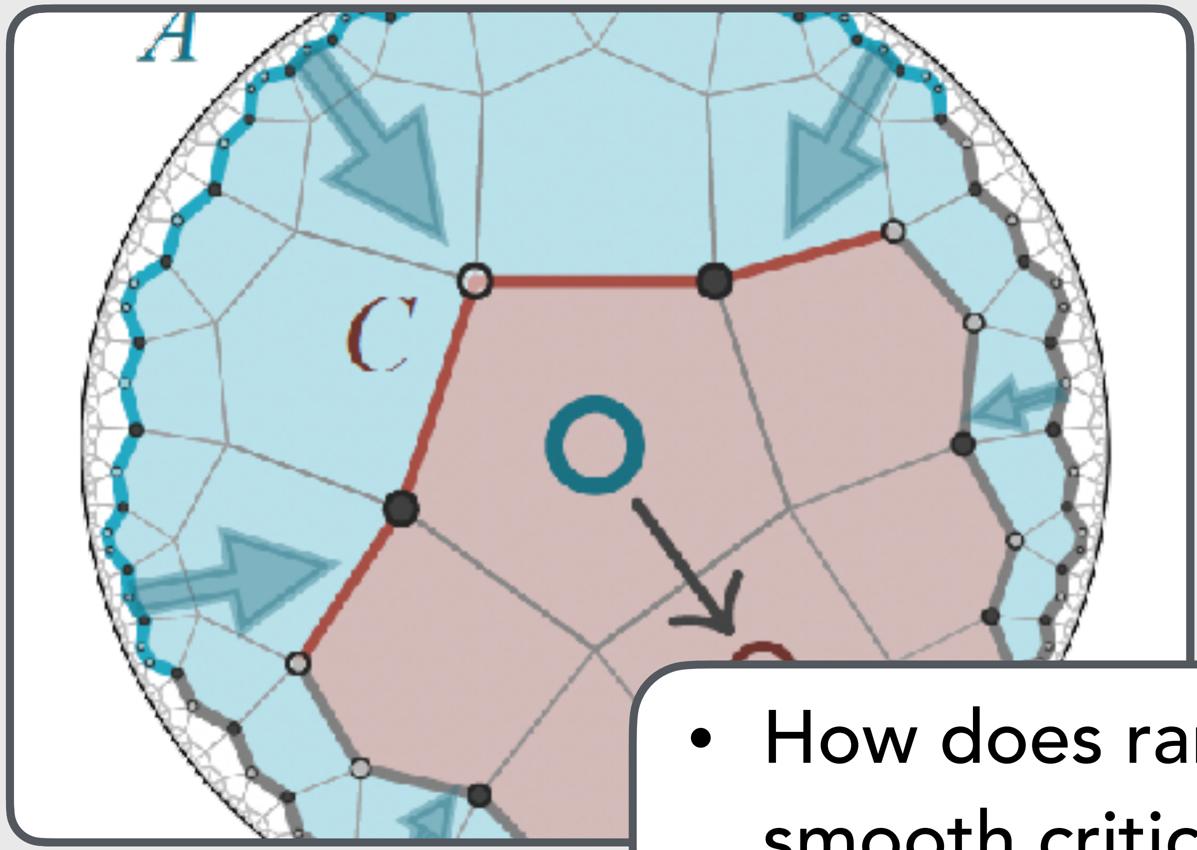
- The complexity is **computationally hard** to compute: How can the notorious conjecture be **proven**?

- Bonus: "Quantum homeopathy" of random quantum circuits

Haferkamp, Faist, Kothakonda, Eisert, Yunger-Halpern, Nature Physics, in press (2022)
Haferkamp, Montealegre-Mora, Heinrich, Eisert, Gross, Roth, arXiv:2002.09524 (2020)

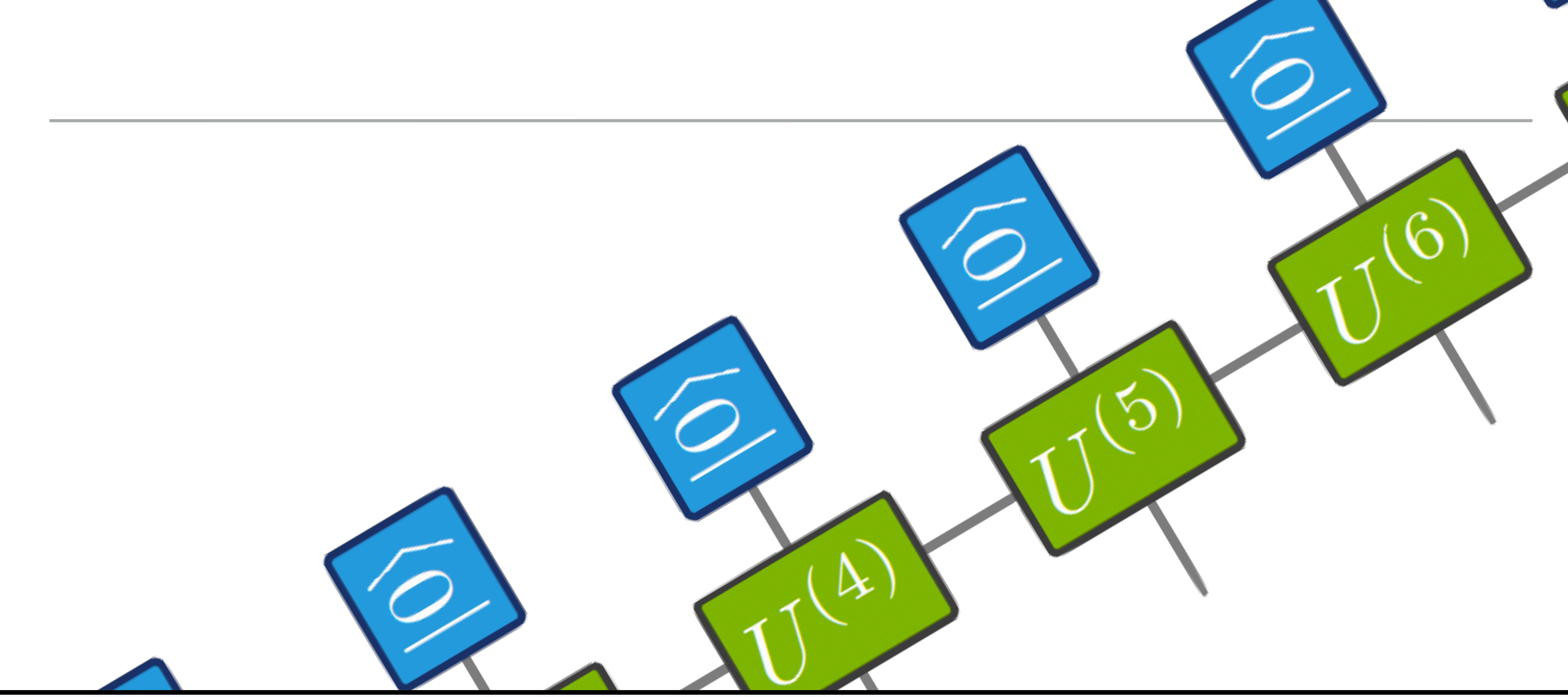
3. RANDOM TENSOR NETWORKS AND HOLOGRAPHY

- Tensor networks as toy models of holography



- How does randomness help to formulate smooth critical conformal field theories?

Jahn, Zimboras, Eisert, Quantum 6, 643 (2022)
Jahn, Eisert, Quant Sc Tech 6, 033002 (2021)
Jahn, Gluza, Pastawski, Eisert, Science Adv 5, eaaw0092 (2019)
Wille, Altland, Jahn, Eisert, in preparation (2022)



STATISTICAL MECHANICS OF RANDOM MATRIX PRODUCT STATES

PRX Quantum 2, 040308 (2021)

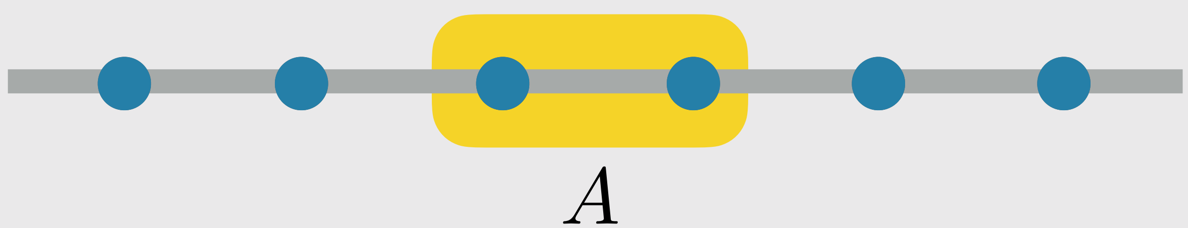
RANDOM MATRIX PRODUCT STATES

- How can **statistical physics** and **quantum dynamics** be reconciled?

von Neumann, Zeitschrift für Physik 57, 30 (1929)
Gogolin, Eisert, Rep Prog Phys 79, 056001 (2016)
Linden, Popescu, Short, Winter, Phys Rev E 79, 061103 (2009)

- Local observables are expected to **equilibrate**

$$|\psi(t)\rangle = e^{-itH}|\psi\rangle$$



- Observables A evolving under H have **time averages**

$$A_\psi^\infty := \lim_{t\to\infty} \frac{1}{t}\int_0^t \langle\psi|A(t')|\psi\rangle \mathrm{d}t'$$

- **Fluctuations** must be small

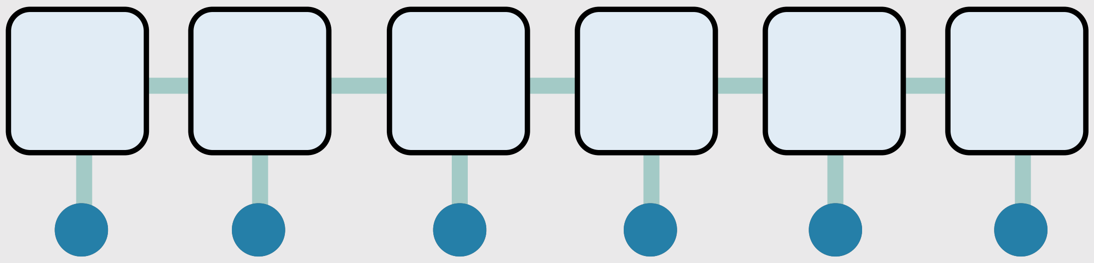
$$\Delta A_\psi^\infty := \lim_{t\to\infty} \frac{1}{t}\int_0^t |\langle\psi|A(t')|\psi\rangle - A_\psi^\infty|^2 \mathrm{d}t'$$

- How can one **judge**?

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)
Lancien, Perez-García, arXiv:1906.11682 (2019)
Garnerone, de Oliveira, Haas, Zanardi, Phys Rev A 82, 052312 (2010)

RANDOM MATRIX PRODUCT STATES

- Random **matrix product states** as initial states*

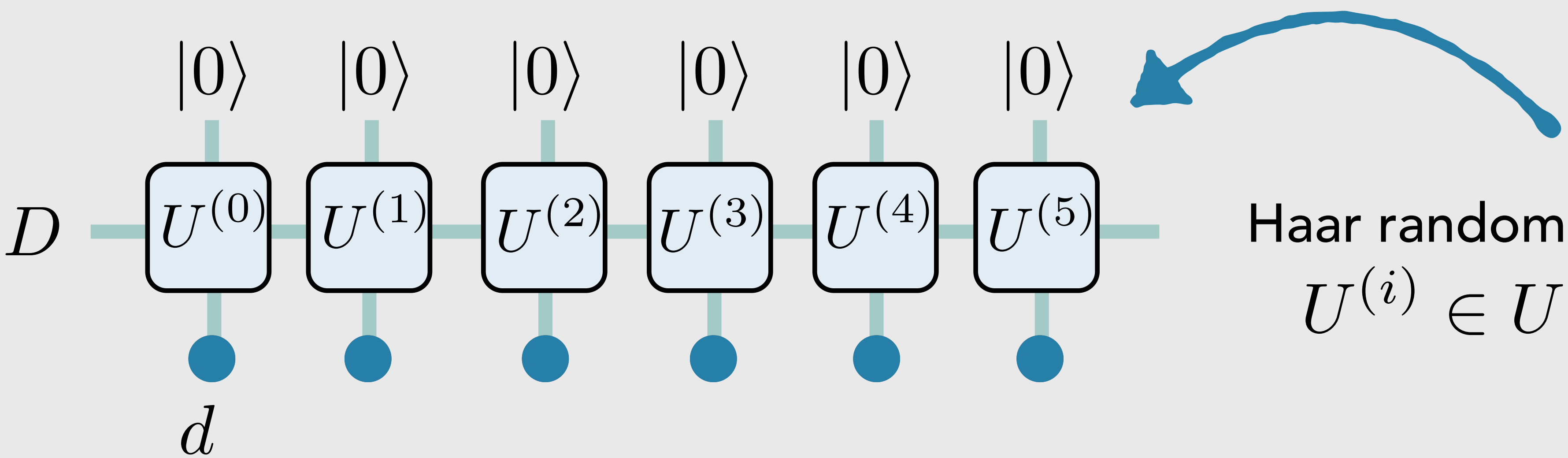


* See Joseph's and Mari-Carmen's tutorials

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)
Lancien, Pérez-García, arXiv:1906.11682 (2019)
Garnerone, de Oliveira, Haas, Zanardi, Phys Rev A 82, 052312 (2010)

RANDOM MATRIX PRODUCT STATES

- Random **matrix product states**

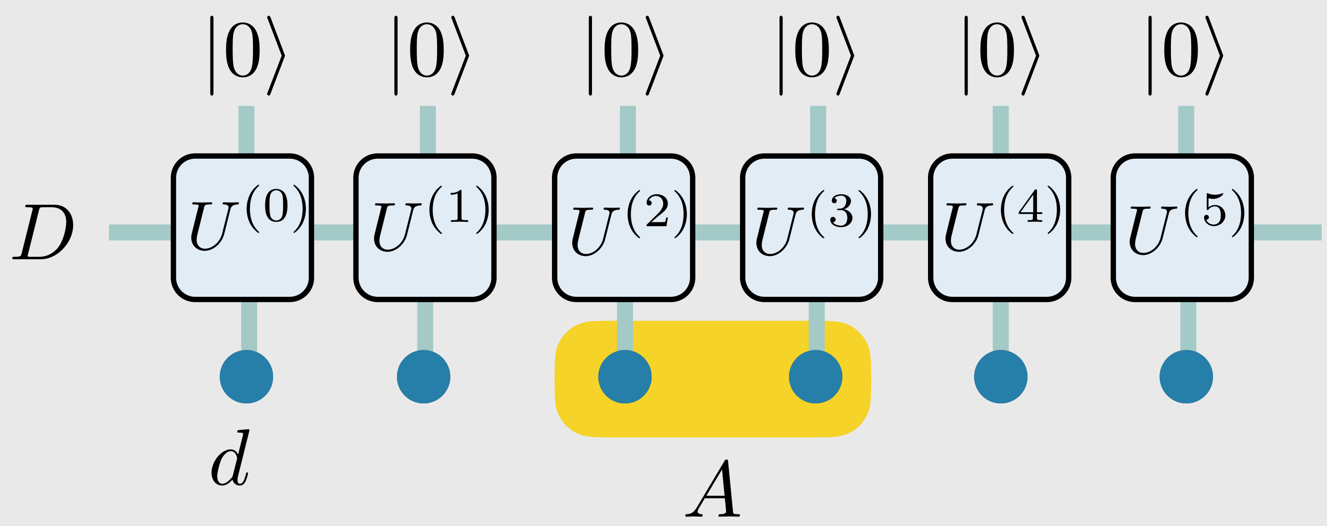


Haar random tensors
$U^{(i)} \in U(dD)$

- Can be seen as generic **representatives of phases of matter**
- Several **interesting properties** can be proven

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)
Lancien, Pérez-García, arXiv:1906.11682 (2019)
Garnerone, de Oliveira, Haas, Zanardi, Phys Rev A 82, 052312 (2010)

- Random **matrix product states**



- They **equilibrate exponentially** well:

$$\Pr\left(\Delta A_{\psi}^{\infty} \leq e^{-c_1 \alpha(d,D) n)}\right) \geq 1 - e^{-c_2 \alpha(d,D) n}$$

for

$$\alpha(d,D) = \log\left(\frac{d - \frac{1}{dD^2}}{(1+\frac{1}{D})(1+\frac{1}{dD})}\right)$$

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)
Compare Huang, Harrow. arXiv:1907.13392 (2019)

IDEA OF PROOF

- Bound "effective dimension"

$$\Delta A_\psi^\infty = O(1/D_{\text{eff}})$$

$$1/D_{\text{eff}} := \sum_j |\langle \psi | j \rangle|^4$$

as overlap of initial state with energy eigenstates

- For any state vector $|\phi\rangle$

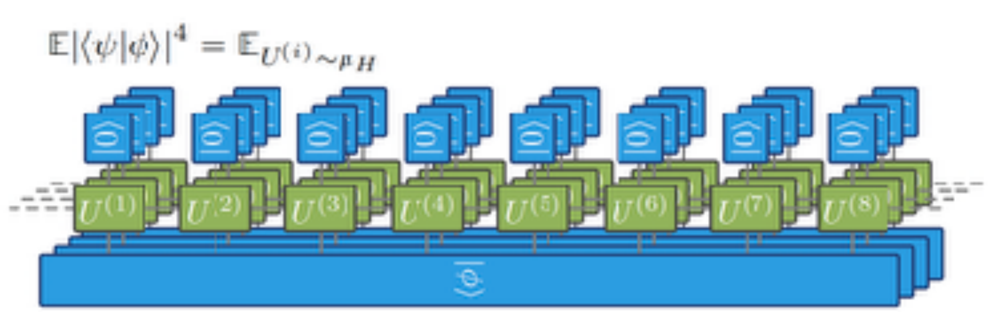
$$\mathbb{E}|\langle\psi|\phi\rangle|^4 = \langle\phi|^{\otimes 2}\mathbb{E}(|\psi\rangle\langle\psi|)^{\otimes 2}|\phi\rangle^{\otimes 2}$$

- The t-th moment operator of Haar-random unitaries

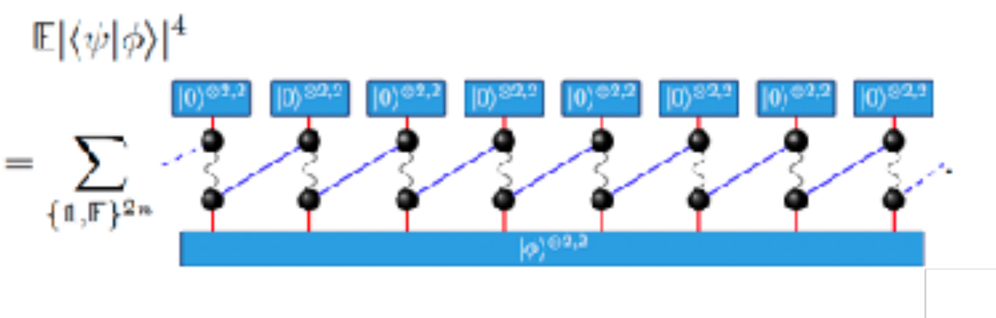
$$\mathbb{E}_{U\sim\mu_H} U^{\otimes t}\otimes \overline{U}^{\otimes t} = \sum_{\sigma,\pi\in S_t} \mathrm{Wg}(\sigma^{-1}\pi, q)|\sigma\rangle\langle\pi|$$

in Weingarten calculus

- Gives tensorial expression

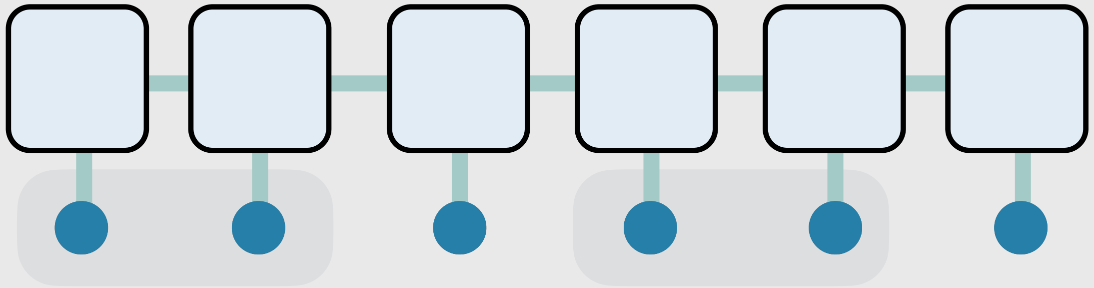


- Map to partition function



Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)
Hunter-Jones, arXiv:1905.12053 (2019)

- Random matrix product states equilibrate exponentially well



- **Further results:**
  - Extensivity of 2-Renyi **entropies**
  - **Maximum entropy** for small connected subsystems
  - Ground states of **disordered parent Hamiltonians**
  - Insights into generic **phases of matter**

  - **Exponential decay of correlations** in similar (TI) model

    Lancien, Perez-García, arXiv:1906.11682 (2019)

Haferkamp, Bertoni, Roth, Eisert, PRX Quantum 2, 040308 (2021)

ENTANGLEMENT IN MANY-BODY SYSTEMS

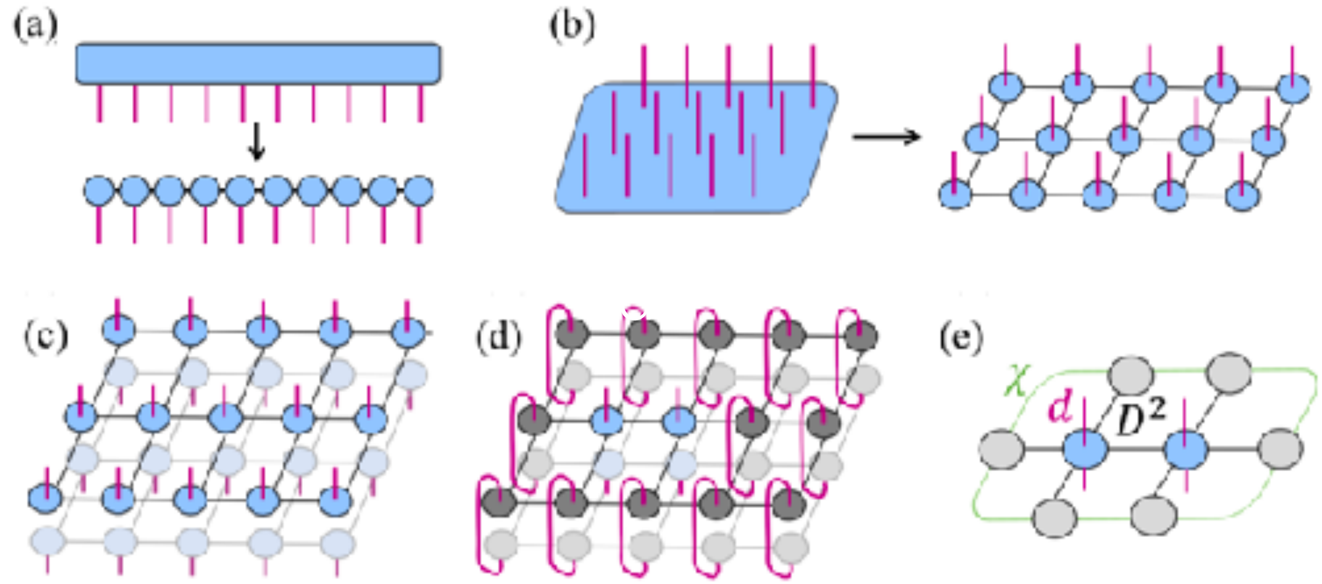
- Noa's poster: Use random sampling to **estimate entanglement** in tensor network-states

- Resource-economically estimate **Renyi entanglement entropies**

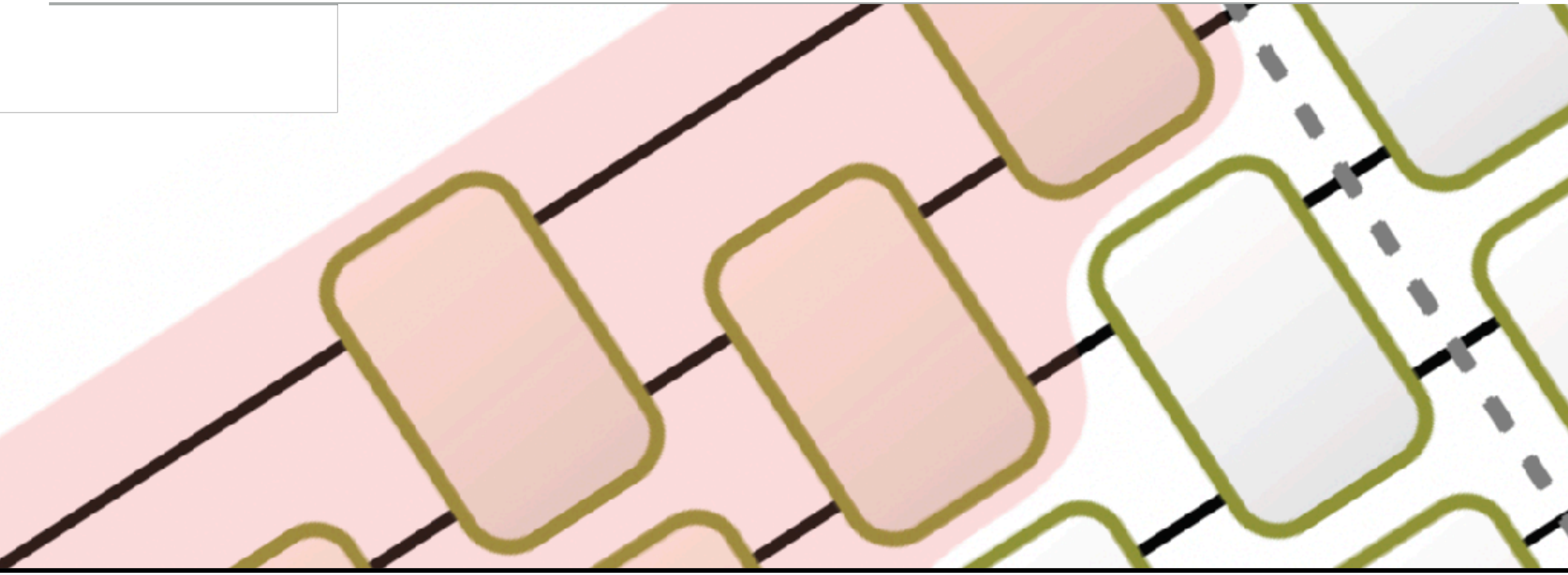
$$E_n(A) = \frac{1}{1-n} \log \mathrm{tr}(\rho_A^n)$$

and negativity moments using **frames**, random vectors $|v\rangle \in \mathbb{C}^d$ with

$$\mathbb{E}(|v\rangle\langle v|) = \mathbb{I}$$



Feldman, Kshetrimayum, Eisert, Goldstein, arXiv:2202.04089 (2022)

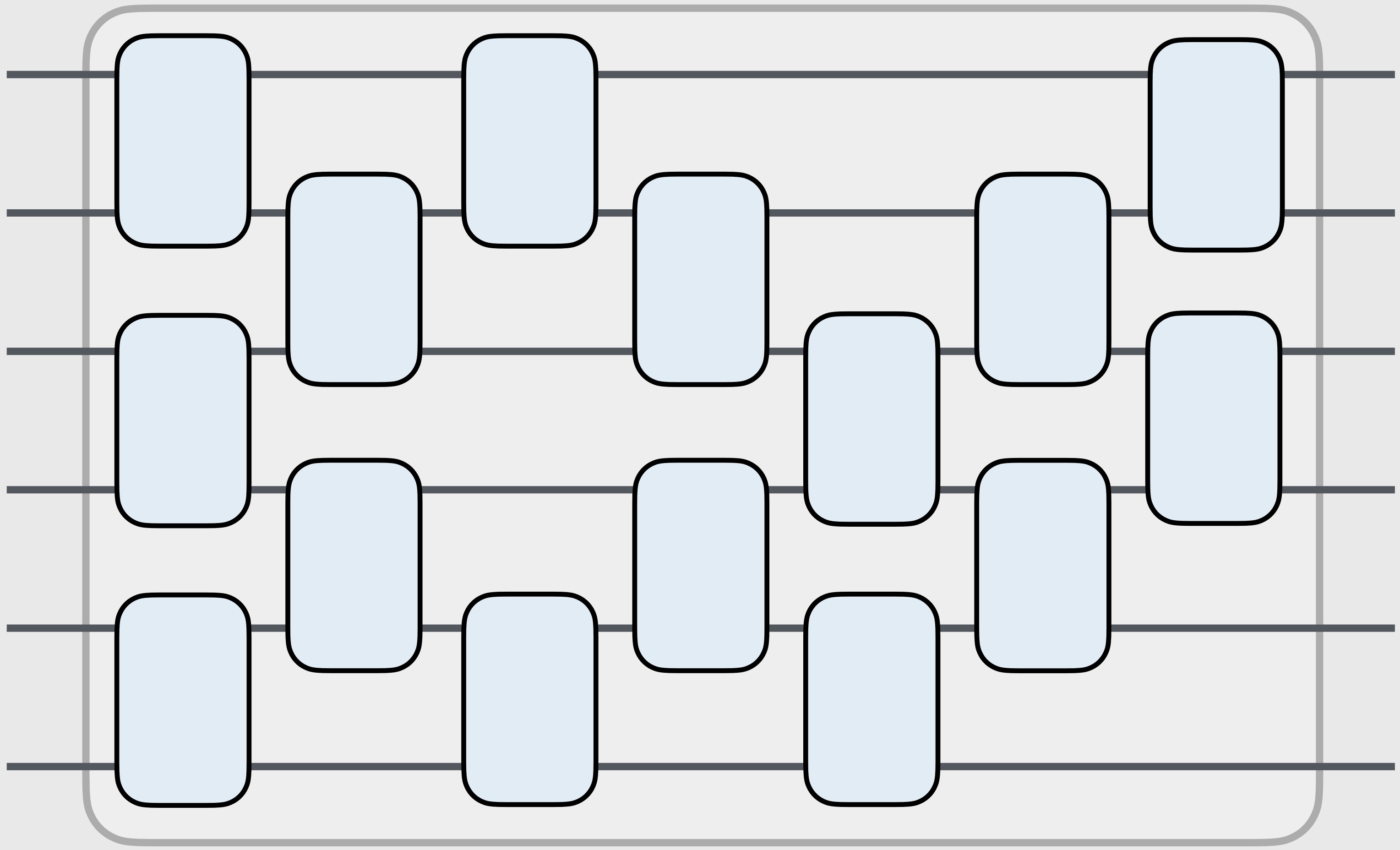


LINEAR COMPLEXITY GROWTH IN RANDOM QUANTUM CIRCUITS

Nature Physics, in press, arXiv:2106.05305 (2022)
Phys Rev Lett 127, 020501 (2021)
arXiv:2002.09524 (2020)

COMPLEXITY GROWTH IN RANDOM CIRCUITS

- **Circuit complexity**: Smallest number of quantum gates from gate set to generate a **given unitary**

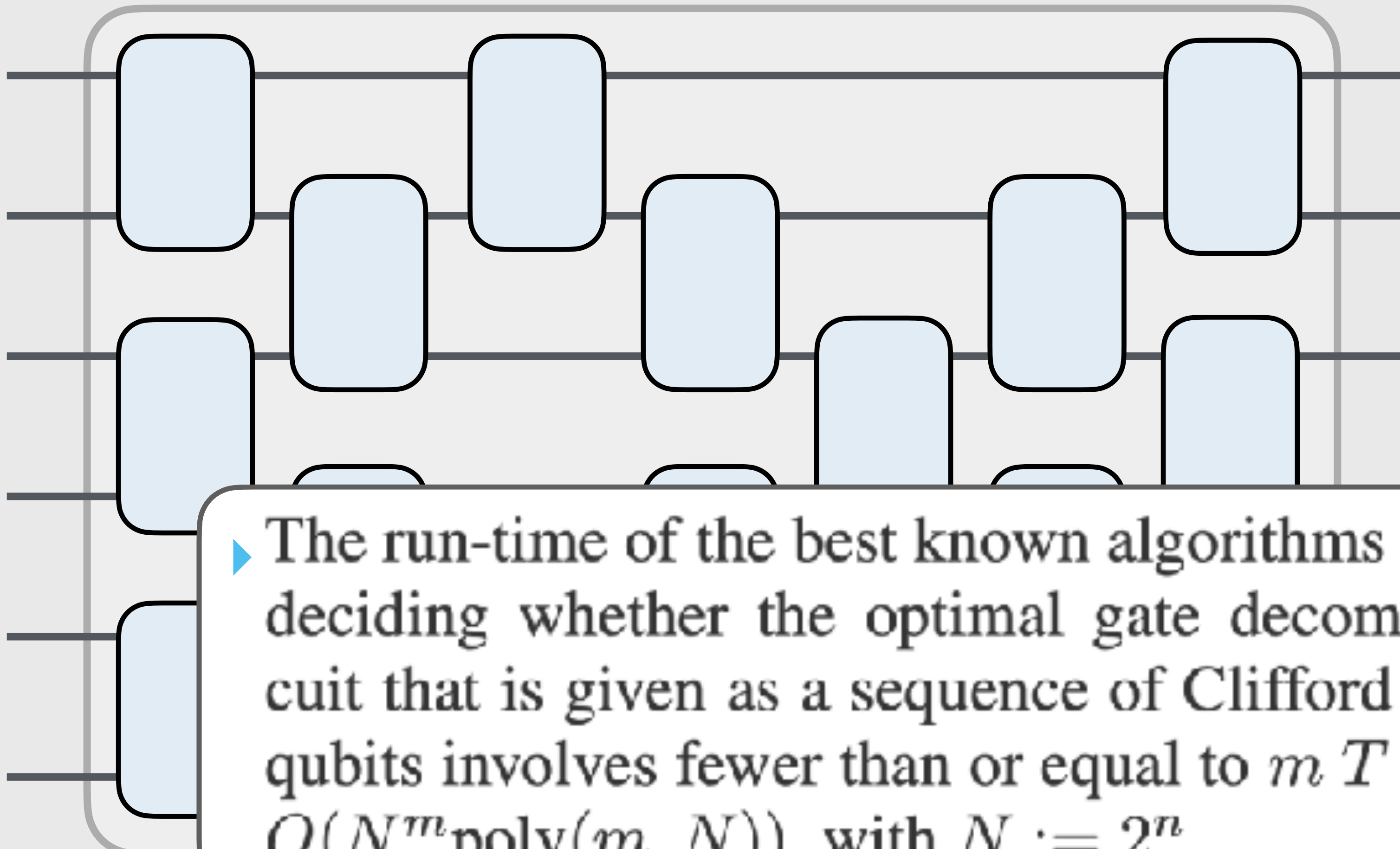


- Separates problems into **'easy'** and **'hard'**
- In quantum setting relevant for **phases of matter**

Gosset, Gosset, Kliuchnikov, Mosca, Russo, Quant Inf Comp 14, 1277 (2014)
Aaronson, Gottesman, Phys Rev A 70, 02328 (2004)

COMPLEXITY GROWTH IN RANDOM CIRCUITS

- **Circuit complexity**: Smallest number of quantum gates from gate set to generate a **given unitary**

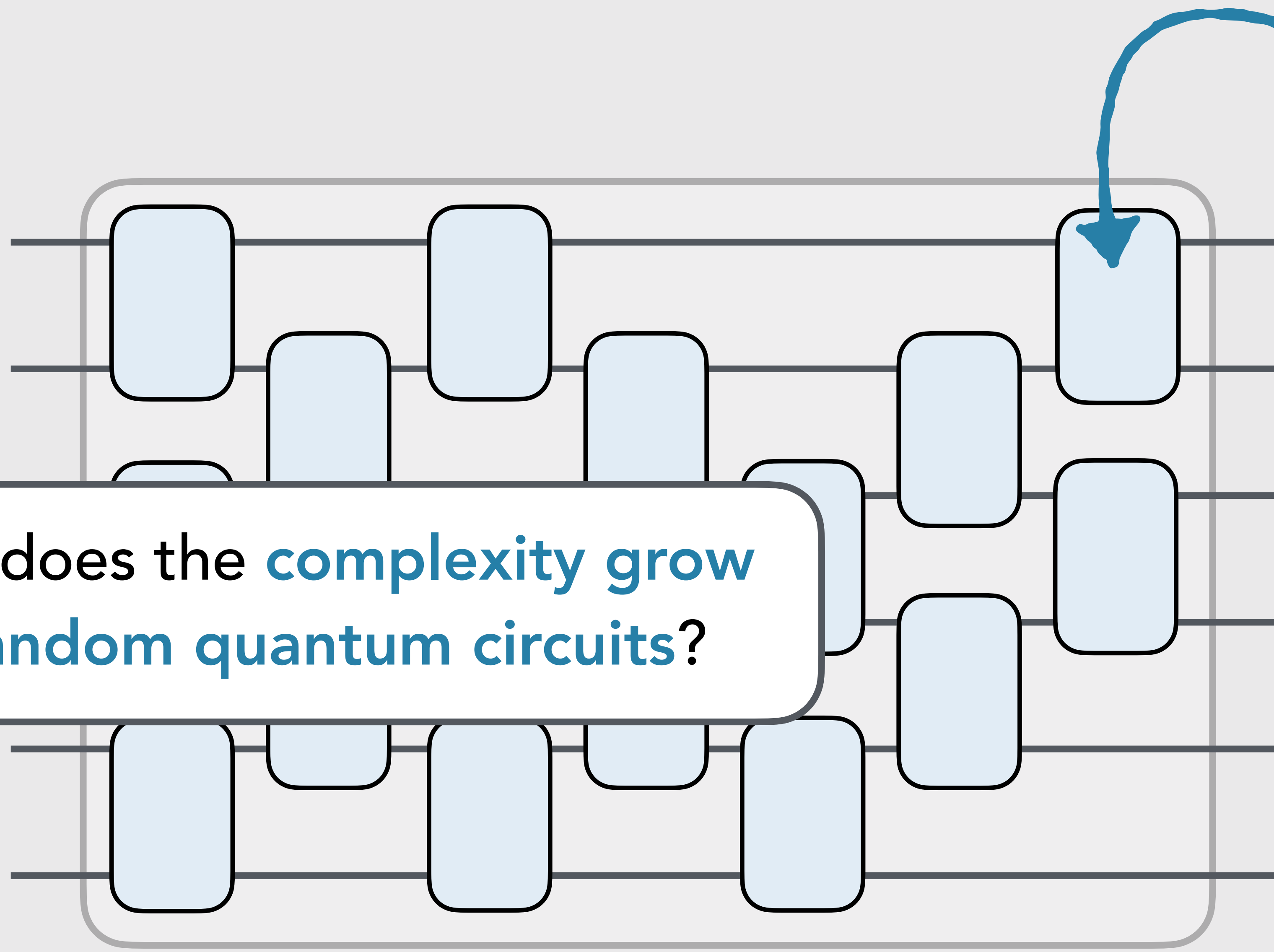


The run-time of the best known algorithms for the T-count—deciding whether the optimal gate decomposition of a circuit that is given as a sequence of Clifford and T gates on n qubits involves fewer than or equal to m T gates or more—is $O(N^m \text{poly}(m, N))$, with $N := 2^n$.

- **Computationally hard:** Notorious cancellations

Gosset, Gosset, Kliuchnikov, Mosca, Russo, Quant Inf Comp 14, 1277 (2014)
Aaronson, Gottesman, Phys Rev A 70, 02328 (2004)

COMPLEXITY GROWTH IN RANDOM CIRCUITS

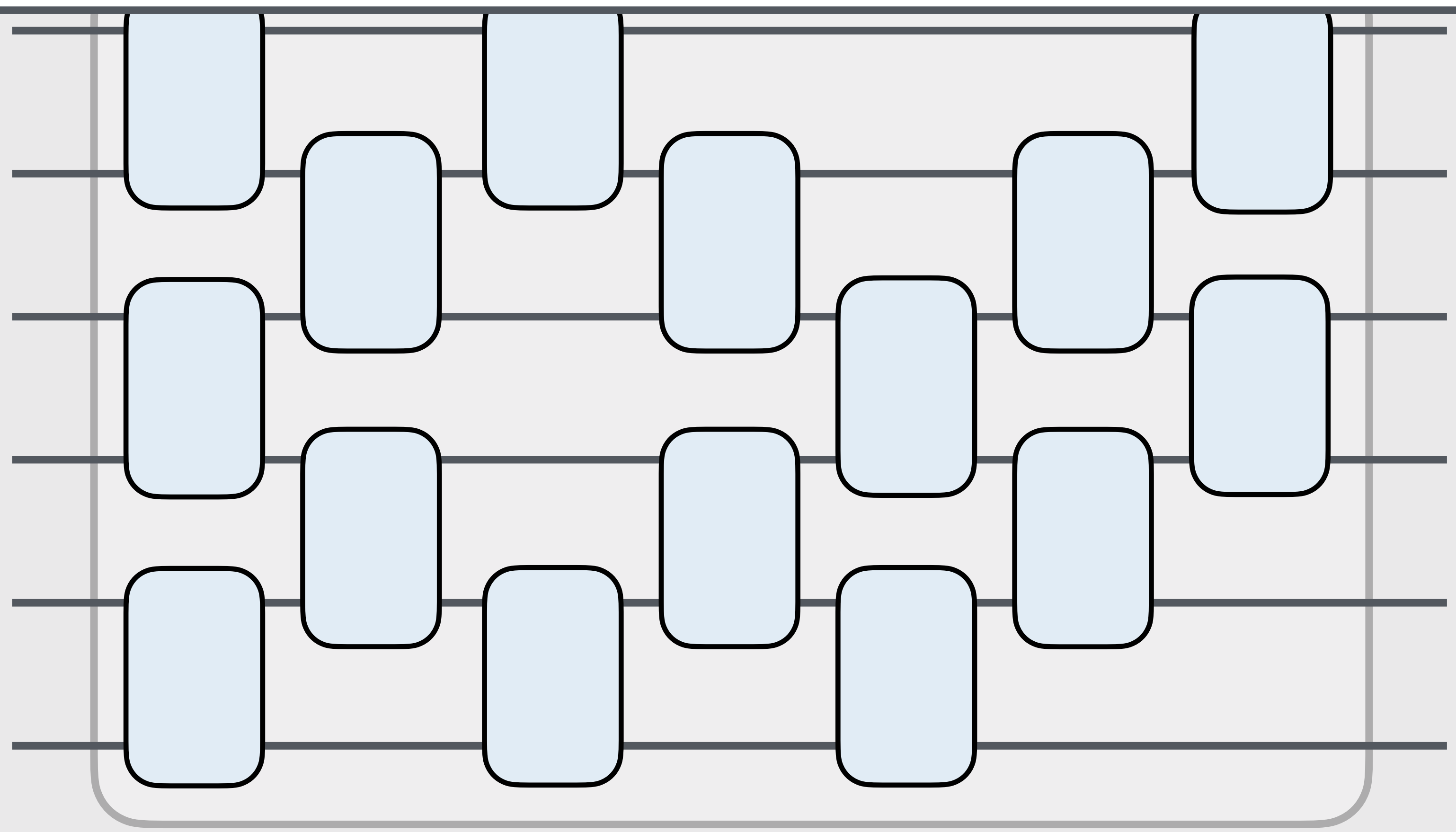


Haar random unitaries, or gates from continuous universal gate set

- How does the **complexity grow** for **random quantum circuits**?

Gosset, Gosset, Kliuchnikov, Mosca, Russo, Quant Inf Comp 14, 1277 (2014)
Aaronson, Gottesman, Phys Rev A 70, 02328 (2004)

- Has risen to prominence as **Brown-Susskind** conjecture

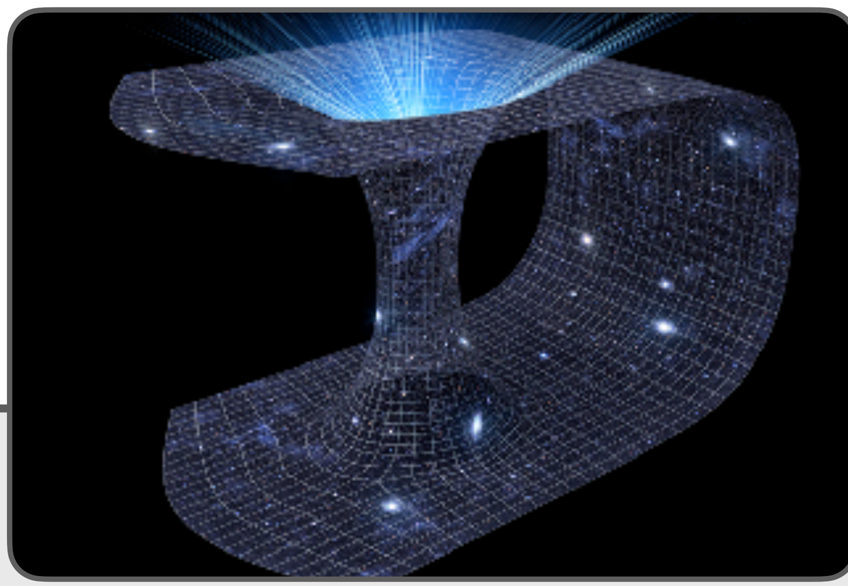


Brown, Roberts, Susskind, Swingle, Zhao, Phys Rev Lett 116, 191301 (2016)
Brown, Roberts, Susskind, Swingle, Zhao, Phys Rev D 93, 086006 (2016)
Chapman, Eisert, Hackl, Heller, Jefferson, Marrochio, Myers, SciPost Phys 6, 034 (2019)
Brown, Susskind, Phys Rev D 97, 086015 (2018)

BROWN-SUSSKIND CONJECTURE

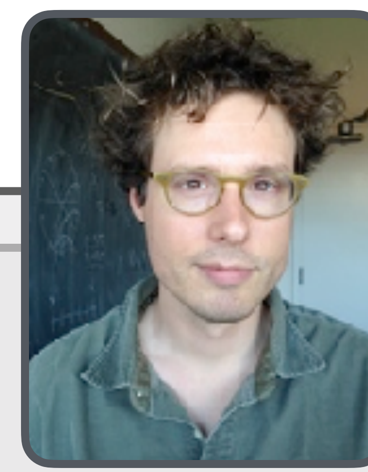
- Has risen to prominence as **Brown-Susskind** conjecture

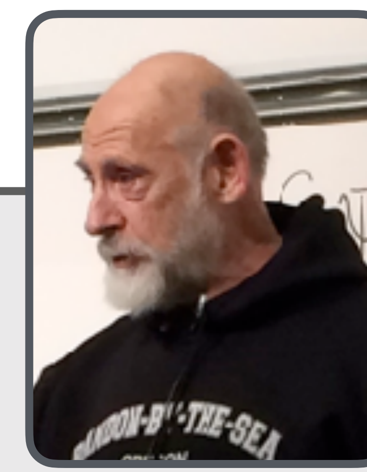
- **AdS:** Volume grows for exponentially long time



- **CFT:** Local observables equilibrating?

$|\psi\rangle$





Brown, Roberts, Susskind, Swingle, Zhao, Phys Rev Lett 116, 191301 (2016)
Brown, Roberts, Susskind, Swingle, Zhao, Phys Rev D 93, 086006 (2016)
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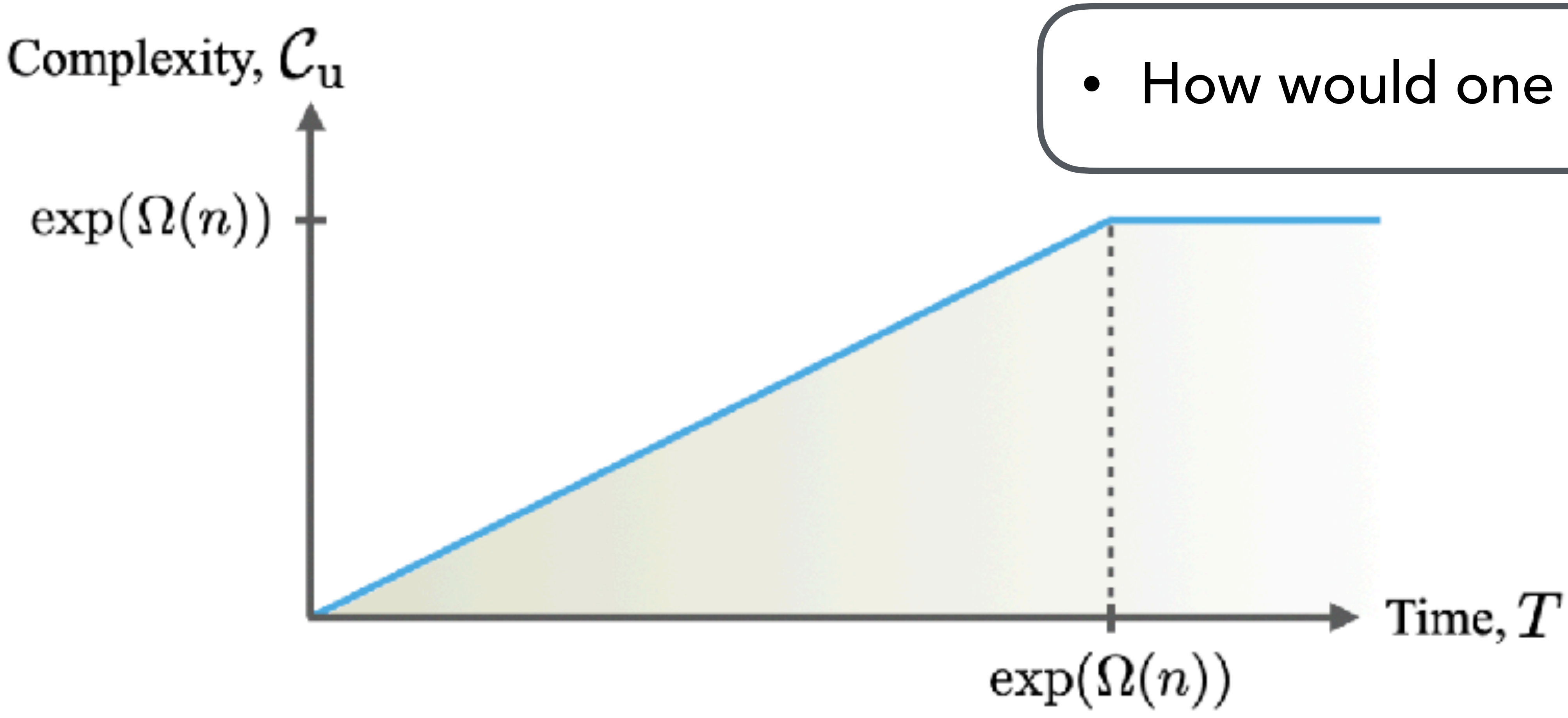
Brown, Susskind, Phys Rev D 97, 086015 (2018)

- Has risen to prominence as **Brown-Susskind** conjecture



- How would one know?

Brown, Susskind, Phys Rev D 97, 086015 (2018)

- Indeed, the linear growth conjecture (until exponential times) is provably **true**!

Haferkamp, Faist, Kothakonda, Eisert, Yunger-Halpern, Nature Physics, in press (2022)

- How can this be judged?

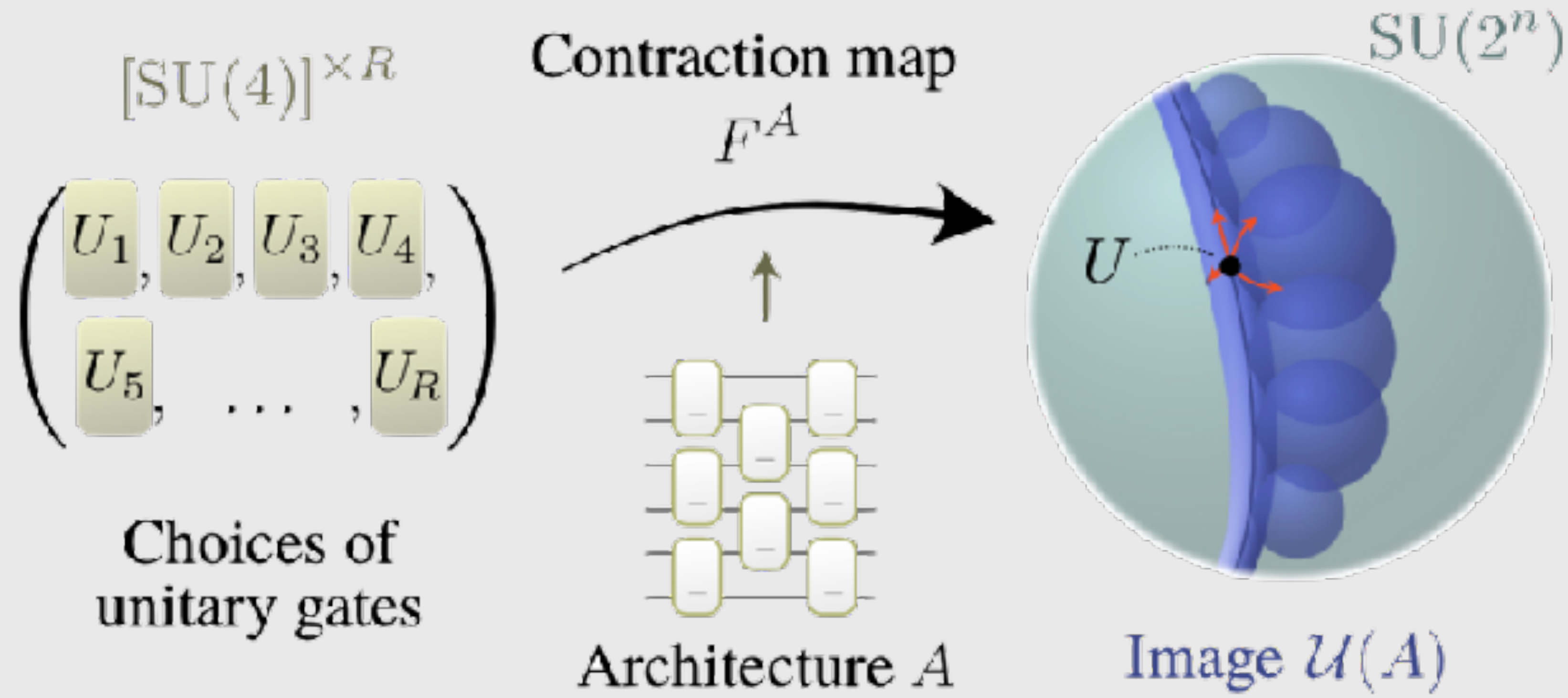
IDEA OF PROOF

- **Contraction map**
  $F^A : \mathrm{SU}(4)^{\times R} \to \mathrm{SU}(2^n)$

- **Quasialgebraic set**: Polynomial equalities and inequalities

- **Tarski-Seidenberg** principle

- **Quasialgebraic set**



- **Accessible dimension*** is almost always the same throughout the domain

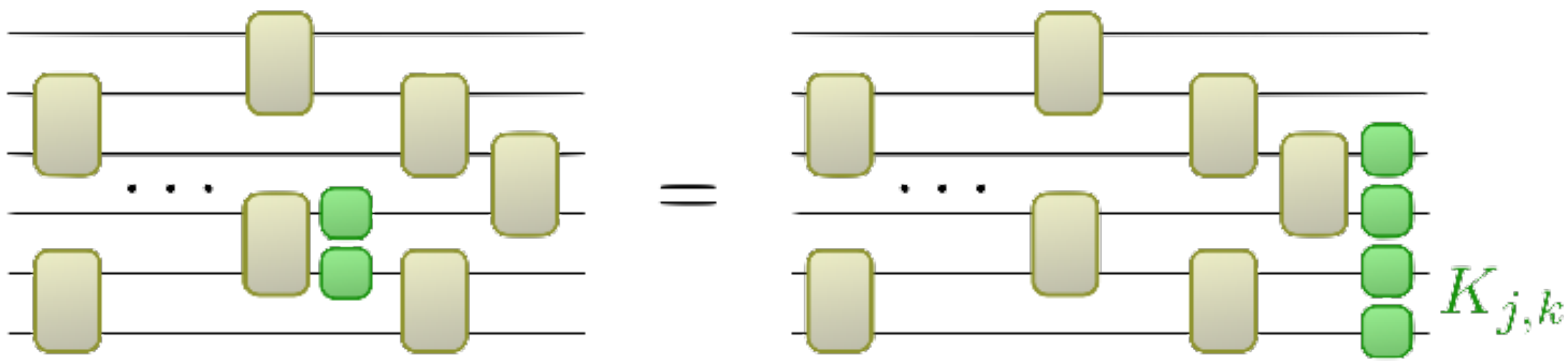
$$d_A = \dim(\mathcal{U}(A))$$

* Set is no manifold

Haferkamp, Faist, Kothakonda, Eisert, Yunger-Halpern, Nature Physics, in press (2022)

IDEA OF PROOF

- Demonstrate the point's existence by perturbing **Clifford circuits,** 'appending infinitesimal unitaries', 'count independent directions'



- Counting

- Identify a point where dimension **grows linearly** with circuit depth

- **Accessible dimension*** is almost always the same throughout the domain

$$d_A = \dim(\mathcal{U}(A))$$

Haferkamp, Faist, Kothakonda, Eisert, Yunger-Halpern, Nature Physics, in press (2022)

- Indeed, the linear growth conjecture (until exponential times) is provably **true**

$$\mathcal{C}_{\mathrm{u}}(U) \geq \frac{R}{9L} - \frac{n}{3} \quad \text{until } T \geq 4^n - 1$$

- Concatenation T blocks involving L gates, involving R unitaries

Haferkamp, Faist, Kothakonda, Eisert, Yunger-Halpern, Nature Physics, in press (2022)

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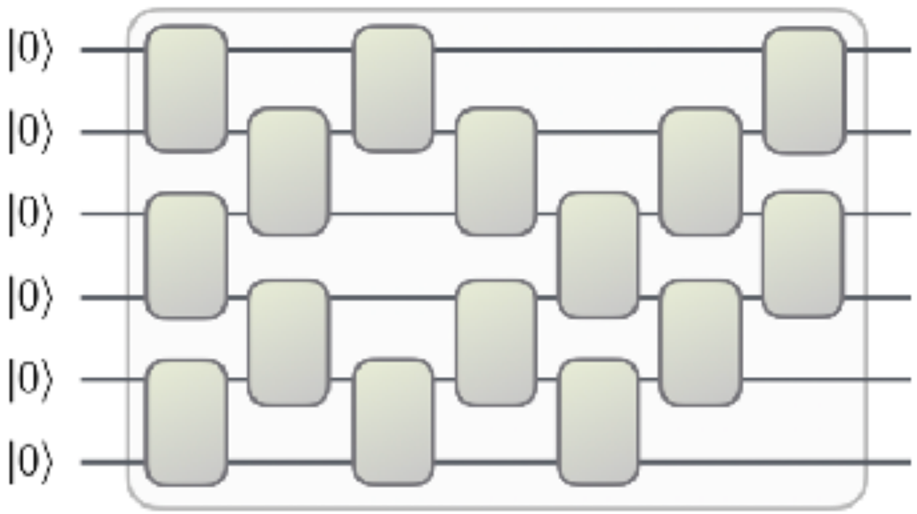
$$\mathcal{C}_{\mathrm{u}}(U) \geq \frac{R}{9L} - \frac{n}{3} \qquad \text{until } T \geq 4^n - 1$$

- **Approximate** versions of **complexity**?
- Connections to **entanglement** and **Nielsen's complexity**?

Nielsen, Dowling, Gu, Doherty, Science 311, 1133 (2006)

- Connect to quantum **chaotic dynamics**?



$$C \geq cE, \; c > 0$$

Eisert, Phys Rev Lett 127, 020501 (2021)

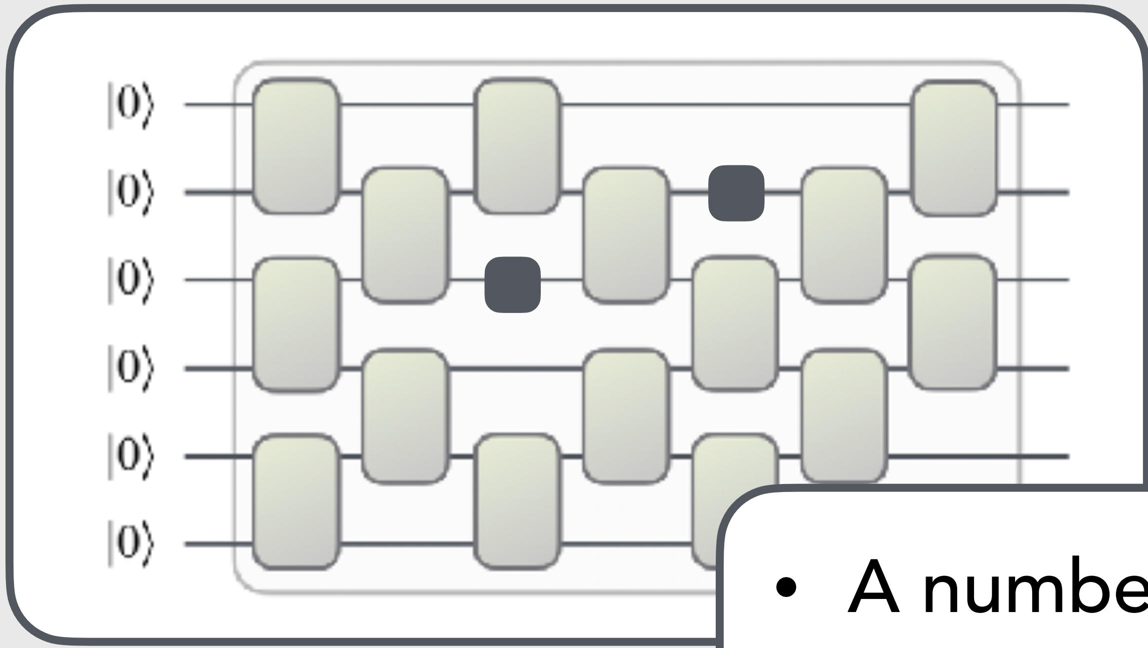
Haferkamp, Faist, Kothakonda, Eisert, Yunger-Halpern, Nature Physics, in press (2022)

WIDER LESSON

- Quantum random circuits are **fun**

- Most **counterintuitive** group result: "**Quantum homeopathy**"
- Random **Clifford circuits** are no (approximate) unitary **4-design**

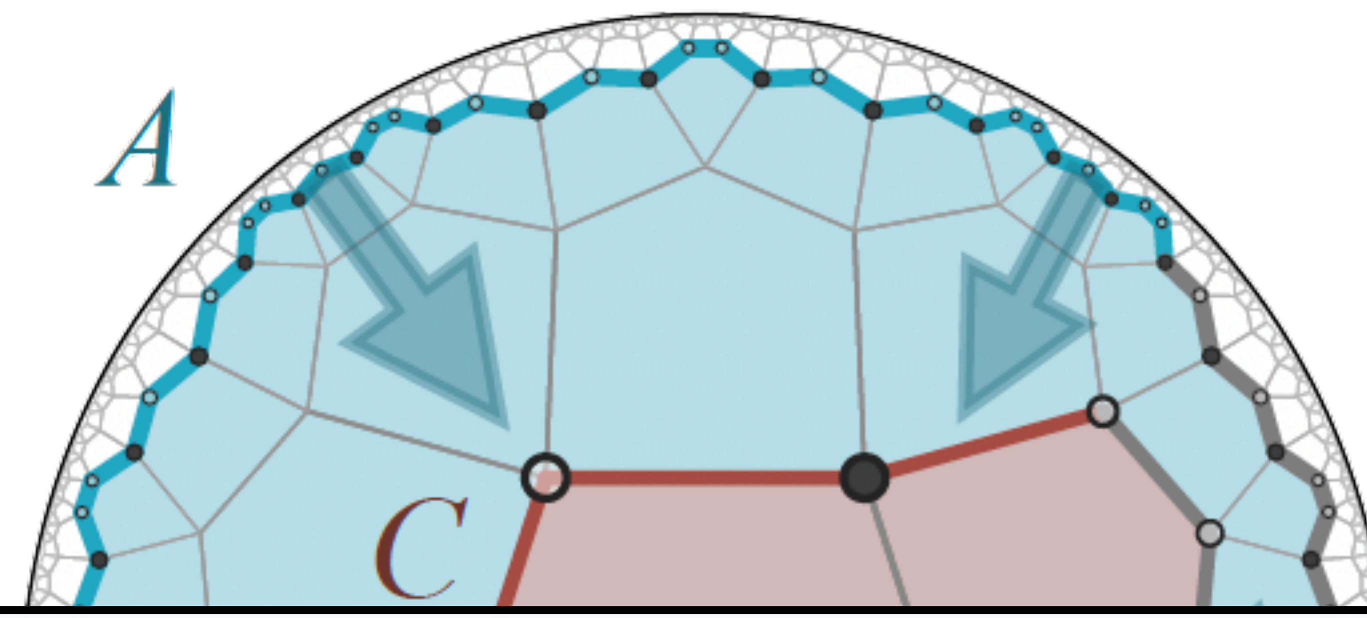
Helsen, Wallman, Wehner, J Math Phys 59, 072201 (2018)
Zhu, Kueng, Grassl, Gross, arXiv:1609.08172 (2016)



- A number of $O(t^4 \log^2(t) \log(1/\varepsilon))$ T-gates uplifts this to an ε approximate t-design

- A **constant** (!) number

Haferkamp, Montealegre-Mora, Heinrich, Eisert, Gross, Roth, arXiv:2002.09524 (2020)

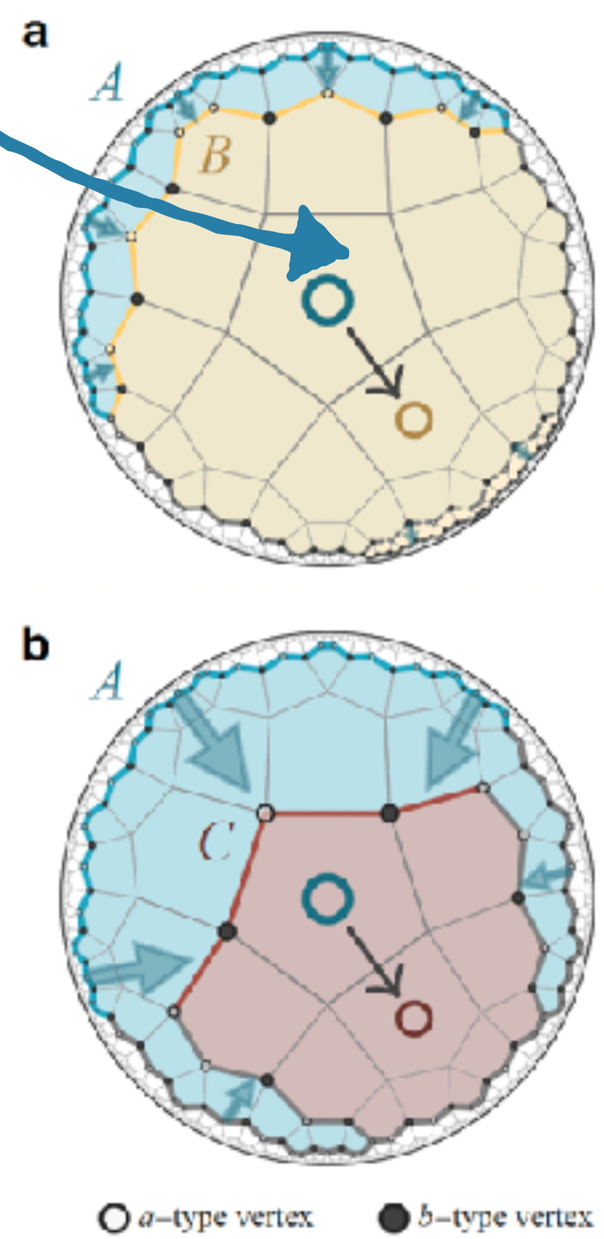


RANDOM TENSORS FOR QUANTUM FIELDS AND HOLOGRAPHY

Quantum 6, 643 (2022)
In preparation (2022)
Quant Sc Tech 6, 033002 (2021)
Science Adv 5, eaaw0092 (2019)

HOLOGRAPHY AND CRITICAL MODELS

Matchgate "free fermionic" **tensor networks** on hyperbolic tiling of plane



- Toy models of AdS-CFT can be formulated as **matchgate tensor networks**

- Can be efficiently (and in instances analytically) contracted

Jahn, Zimboras, Eisert, Quantum 6, 643 (2022)
Jahn, Eisert, Quant. Sc. Tech. 6, 033002 (2021)
Wille, Altland, Jahn, Eisert, in preparation (2022)

HOLOGRAPHY AND CRITICAL MODELS

Matchgate "free fermionic" tensor networks on hyperbolic tiling of plane

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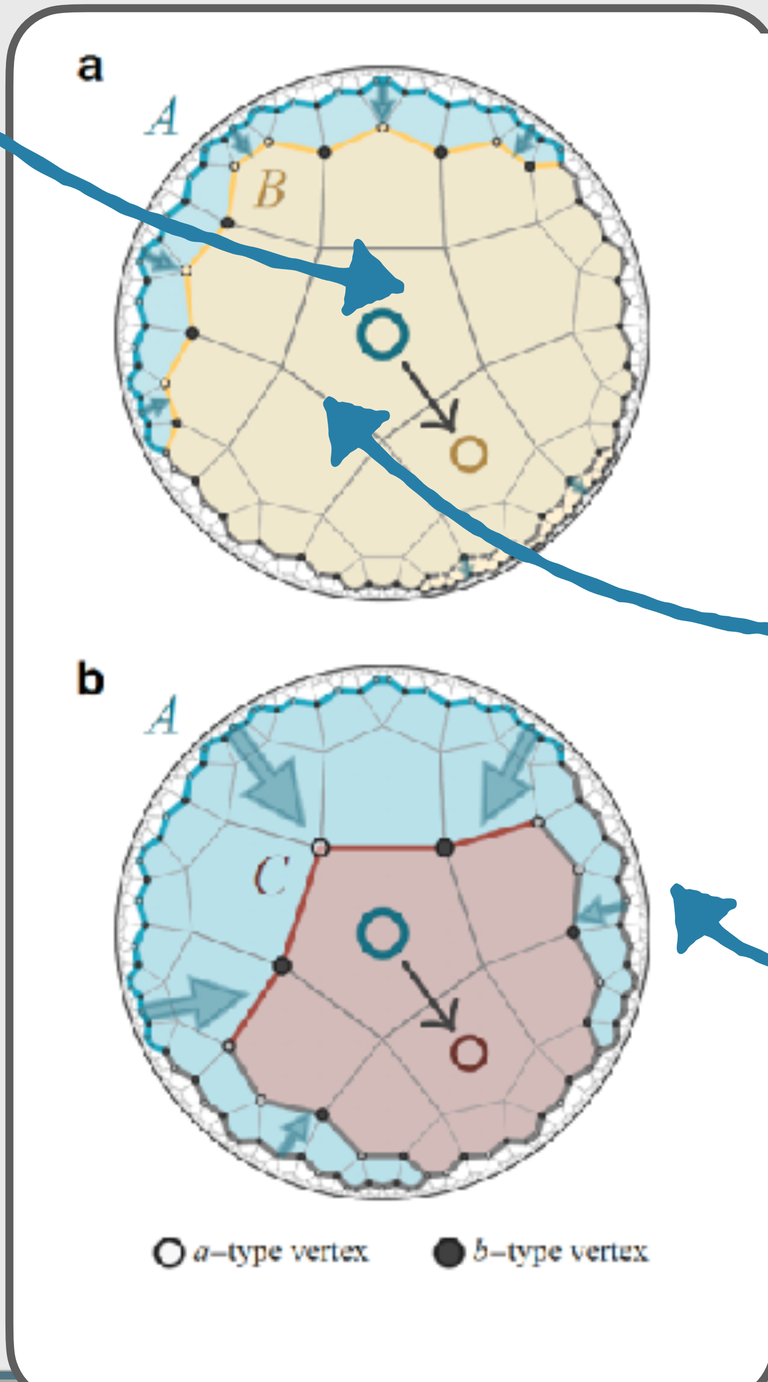


Inflation rules to go from one layer to the next

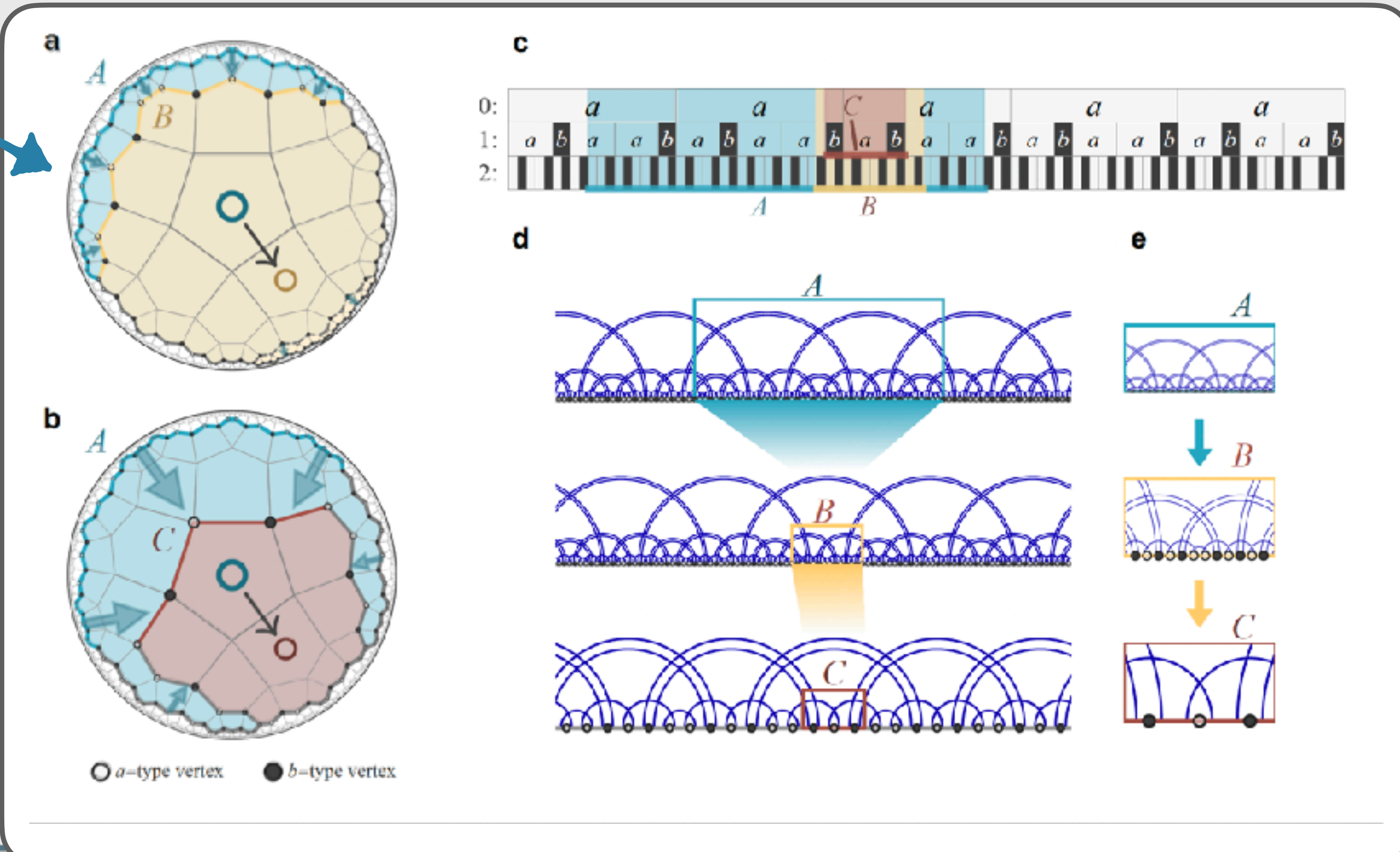
Critical theory on boundary with effective central charges depending on tiling, e.g.

$$c_{\{5,4\}} \approx 4.74$$

Jahn, Zimboras, Eisert, Quantum 6, 643 (2022)
Jahn, Eisert, Quant. Sc. Tech. 6, 033002 (2021)
Wille, Altland, Jahn, Eisert, in preparation (2022)

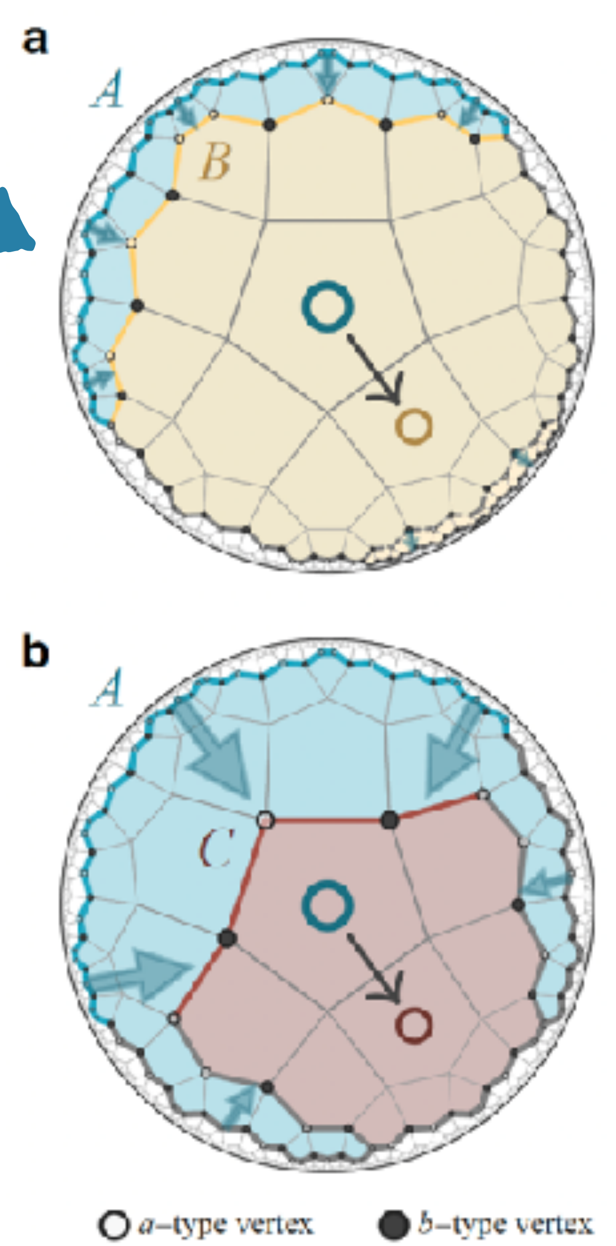
- Toy models of AdS-CFT can be formulated as **matchgate tensor networks**

Get actual CFT (up to quasi-crystalline symmetry)



HOLOGRAPHY AND CRITICAL MODELS

Get **actual CFT** (up to quasi-crystalline symmetry)



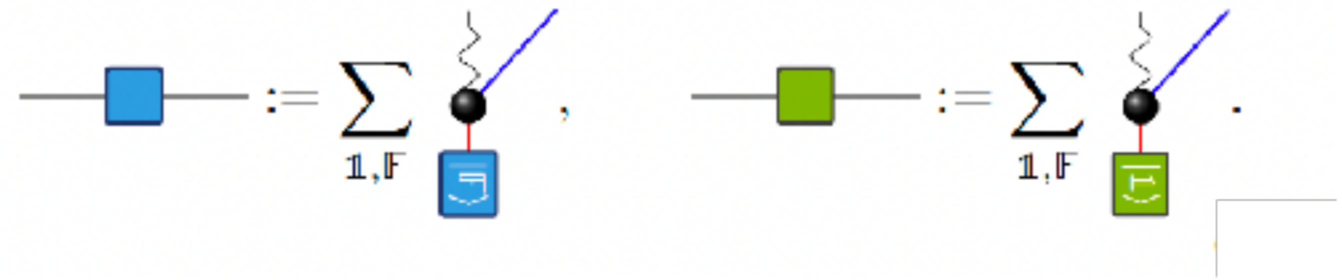
- Using **random matchgate tensors**, can go to the continuum (in prep)

Jahn, Zimboras, Eisert, Quantum 6, 643 (2022)
Jahn, Eisert, Quant Sc Tech 6, 033002 (2021)
Wille, Altland, Jahn, Eisert, in preparation (2022)

OUTLOOK

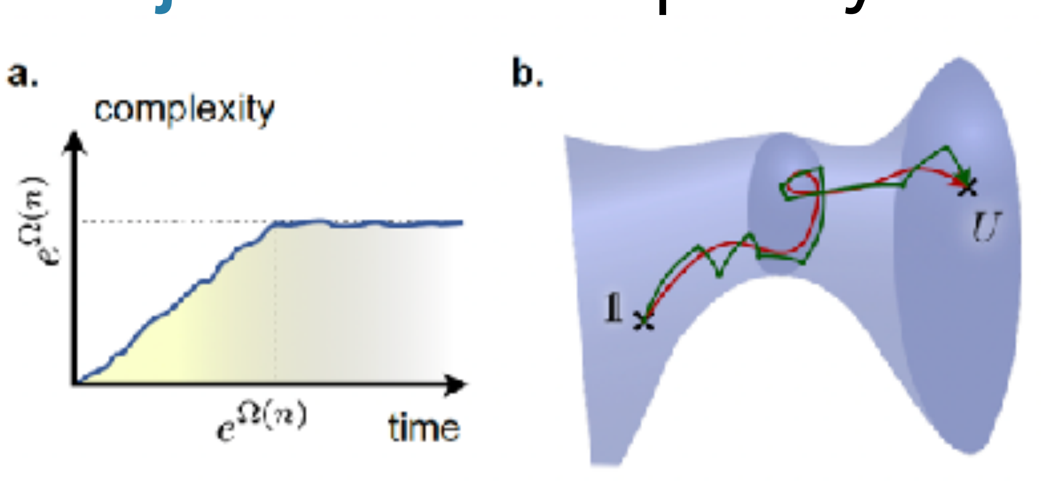
OUTLOOK

- Statistical mechanics of random tensor networks



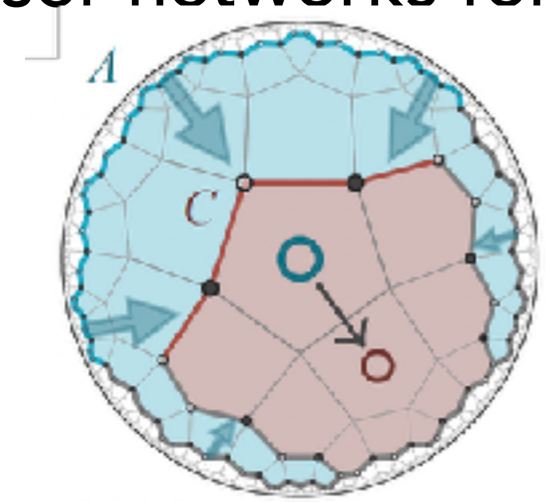
- More "physical" probability measures?

- Solving the Brown Susskind conjecture of complexity



- Approximate notions?

- Tensor networks for holography



- Get full conformal field theories

- So, how much randomness do we need in random tensor networks?

THANKS FOR YOUR ATTENTION!

POSTDOC AVAILABLE