Generalized strong convergence in iid random unitaries

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Overview

Joint work with Charles Bordenave (CNRS Marseille)

Plan:

1. Asymptotic strong freeness: statement of the result.
2. Operator valued non-backtracing theory.
3. Centered Weingarten calculus.
4. Random Unitary vs GUE.

NC distribution and convergences

- Let $X_1^{(n)}, \dots, X_d^{(n)}$ be elements of a tracial NCPS $(A^{(n)}, \tau^{(n)})$ and $X_1, \dots, X_d$ be elements of a tracial NCPS (A, τ). Convergence in NC distribution holds iff for any NC polynomial P in d variables and its adjoint,

$$\tau^{(n)} P(X_i^{(n)}) \to \tau P(X_i)$$

- If, in addition,

$$||P(X_i^{(n)})|| \to ||P(X_i)||,$$

  then one speaks of strong convergence

NC distribution and convergences

- In general, $(A^{(n)}, \tau^{(n)})$ are matrix algebras.
- When the traces are faithful,

$$\liminf_n ||P(X_i^{(n)})|| \geq ||P(X_i)||$$

  always holds
- The inequality can be strict, e.g. take $X_i = U_i \otimes \overline{U}_i$ where U_i are LPS (or expander) generators acting on an irrep.

Asymptotic freeness

- If the limiting objects $U_1, \ldots, U_d$ are $*$-free then one speaks of asymptotic freeness or strong asymptotic freeness.
- Important examples of asymptotic freeness were given by Voiculescu: iid GUE copies, or iid Haar distributed unitaries of dimension n.
- Other examples by Nica (random permutations), Biane (truncated Jucys Murphy elements).

Strong Asymptotic freeness

- The first series of examples of strong asymptotic freeness were obtained by Haagerup-Thorbjørnsen: iid GUE (2005). This was extended in many directions (Capitaine, Donati, Male, ...).
- The second class of examples was obtained by C & Male: iid Haar random unitary matrices (2012). Remark: a quantitative version was obtained by F. Parraud with free stochastic calculus.
- More recently, Bordenave and C obtained similar results for random permutations (2018).

Strong Asymptotic freeness: main result

Theorem (Bordenave, C – 2020)

$(\overline{U}_i^{\otimes q_-} \otimes U_i^{\otimes q_+})_{i=1,\ldots,d}$ are strongly asymptotically free as $n \to \infty$ on the orthogonal of fixed point spaces.

The same holds true for random orthogonal matrices.

Corollary

For any non-trivial signature (λ, ρ), consider for n large enough (when defined) the quotient image of $\mathbb{U}_n$ under the irreducible representation $\mathbb{U}(V_{\lambda,\rho,n})$. Take d iid copies according to the Haar measure. They are strongly asymptotically free as $n \to \infty$.

Main result: comments

- The corollary is a zero-one law. It tells (in this context) that the only obstruction for strong freeness are the fixed points. This sheds light on the important counterxample $U_i \otimes \overline{U}_i$.
- The corollary follows from the main theorem because every irrep is contained in a tensor representation of the theorem.

Main result: geometric considerations

- The result is not surprising: moments converge faster to zero for tensors (so, asymptotic freeness is not difficult).
- But the evaluation of the operator norm is hard: same amount of randomness, but sup to be taken over a much larger space (so, strong asymptotic freeness is much harder).

Main result: geometric considerations

- For 'regular' unitary matrices, the size of an eta-net of vectors is $(C/\eta)^n$ and the concentration speed is $\exp(-c\epsilon^2 n)$.
- In our model, concentration speed does not change but the eta-net becomes $(C/\eta)^{n^q}$ so it becomes impossible to obtain strong convergence 'up to a universal constant' by soft methods.

Main result: analysis vs combinatorics

- Although asymptotic freeness results can be obtained by moment computations, strong asymptotic freeness results until 2018 all relied on analysis: linearization, analytic properties of matrix valued Stieltjes transform (IBP - Schwinger-Dyson - loop equation), complex analysis: moments methods were too bulky to implement.
- However, Bordenave and C's results for random permutations (2018) rely on moments through new techniques.
- Here we need to expand these techniques.

Outline of the proof

- As in Bordenave-C-2018, we need to evaluate the norm of

$$\sum_{i=-d}^{d} a_i \otimes X_i^{(n)},$$

  where $X_{-i}^{(n)} = X_i^{(n)*}$ and $X_0^{(n)} = Id$. [linearization step]
- Even this simplified object is too hard to evaluate with moments. We replace it by an operator valued non-backtracking matrix.

Operator valued non-backtracking theory

We consider $(b_1, \ldots, b_l)$ elements in $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is a Hilbert space. We assume that the index set is endowed with an involution $i \mapsto i^*$ (and $i^{**} = i$ for all i).
The non-backtracking operator associated to the ℓ-tuple of matrices $(b_1, \ldots, b_l)$ is the operator on $\mathcal{B}(\mathcal{H} \otimes \mathbb{C}^l)$ defined by

$$B = \sum_{j \neq i^*} b_j \otimes E_{ij}, \quad (1)$$

Left non-backtracking operator:

$$\widetilde{B} = \sum_{j \neq i^*} b_i \otimes E_{ij}. \quad (2)$$

Operator valued non-backtracking theory

Theorem

Let $\lambda \in \mathbb{C}$ *satisfy* $\lambda^2 \notin \{spec(b_i b_{i^*}) : i \in [[l]]\}$. *Define the operator* A_λ *on* $\mathcal{H}$ *through*

$$A_\lambda = b_0(\lambda) + \sum_{i=1}^{\ell} b_i(\lambda)\,, \qquad b_i(\lambda) = \lambda b_i(\lambda^2 - b_{i^*} b_i)^{-1}$$

and

$$b_0(\lambda) = -1 - \sum_{i=1}^{\ell} b_i(\lambda^2 - b_{i^*} b_i)^{-1} b_{i^*}\,.$$

Then $\lambda \in \sigma(B)$ *if and only if* $0 \in \sigma(A_\lambda)$.

Operator valued non-backtracking theory

- In practice we just have to understand the spectral radius of the operator and therefore, evaluate $\tau(B^T B^{*T})$ with T growing with the matrix dimension.
- The non backtracking structure makes calculations tractable... through Weingarten calculus.
- This answers a question by Pisier (prove a variant of HT in the unitary setup with moment methods).

Centered Weingarten calculus

- For a random variable X, we define $[X] = X - E(X)$ (its centering).
- For a symbol $\varepsilon \in \{\cdot, -\}$ and $z \in \mathbb{C}$, we take the notation that $z^{\varepsilon} = z$ if $\varepsilon = \cdot$ and $z^{\varepsilon} = \overline{z}$ if $\varepsilon = -$. We want to to compute for $U = (U_{ij})$ Haar distributed on $\mathbb{U}_n$, expresssions of the form

$$E \prod_{t=1}^{T} [\prod_{l=1}^{k_t} U_{x_{tl} y_{tl}}^{\varepsilon_{tl}}]$$

in a meaningful way.

Centered Weingarten calculus

- We can write a Weingarten formula

$$E\prod_{t=1}^{T}[\prod_{l=1}^{k_t} U_{x_{tl}y_{tl}}^{\varepsilon_{tl}}] = \sum_{\sigma,\tau\in P_2(k_1+\ldots+k_T)} \delta_{\sigma,x}\delta_{\tau,y} Wg(\sigma,\tau,part)$$

- The function Wg depends on the pairings and the partition.

Theorem

Wg decays as n^{-k} where
$k = (k_1+\ldots+k_T)/2 + d(\sigma,\tau) + 2\#lonesome\ blocks.$

- This estimate is uniform on $k \sim Poly(n)$.

Comparison with iid gaussians: problem

- We want to compare $\sqrt{n}U_{ij}$ with iid complex gaussian matrices $(G_{ij})_{i,j\in[[k]]}$.
- Known results:
  Convergence in total variation distance for $k << \sqrt{n}$ (Olshanski, C, Jiang).
  Uniform convergence of joint moments of order $k << n^{2/7}$ (C, Matsumoto).

Comparison with iid gaussians: why?

- Observation (Bordenave, C): if the centered moments of $\sqrt{n}U_{ij}$ and $(G_{ij})_{i,j\in[[k]]}$ have a uniformly equivalent asymptotic behavior then we can conclude.
- We use the fact that there exists a Wick calculus for centered gaussian moments (sum over non-lonesome pairings)

Comparison with iid gaussians

Technical result:
Let k even with $2k^{7/2} \leqslant n^2$ and $n \geqslant 4$, $\pi = (\pi_t)_{t\in[[T]]}$ be a partition of $[[k]]$ such that each block contains at most l elements. Let $\varepsilon \in \{\cdot, -\}^k$ be a balanced sequence. For any x, y in $[[n]]^k$, we have

$$n^{k/2}\left|E(\prod_{t=1}^{T}[\prod_{i\in\pi_t} U_{x_i y_i}^{\varepsilon_i}])\right| \leq (1+\delta)E(\prod_{t=1}^{T}([\prod_{i\in\pi_t}(G_{x_i y_i}^{\varepsilon_i}+\eta_1)]+\eta_2)),$$

with $\delta = 3k^{7/2}n^{-2}$, $\eta_1 = l^{1/2}k^{7/8}n^{-1/2}$ and $\eta_2 = k^{\ell}n^{-1/4}$.
Moreover, if each block π_t contains an element with $\varepsilon_i = \cdot$ and another with $\varepsilon_i = -$, the same bound holds with $\eta_2 = k^l n^{-1/2}$.

Thank you!