Nice Error Basis & Study of Quantum Maps

Purbayan Chakraborty* (Joint work with Uwe Franz** and B.V.R. Bhat***)

* Laboratoire de Mathématiques de Besançon, Université Franche Comté, Email Id: purbayan.chakraborty@univ-fcomte.fr

** Laboratoire de Mathématiques de Besançon, Université Franche Comté

*** Stat-Math Unit, Indian Statistical Institute Bangalore

Introduction

Nice error bases (NEB) are generalisation of Pauli matrices in higher dimesnion. It is very important for quantum information theory e.g. quantum error correcting code, teleportation etc. as they form very "nice" bases of the matrix algebra $M_n(\mathbb{C})$.

In this work we start with a NEB and further construct a NEB of the space $L(M_n, M_n)$ - the space of all linear maps between M_n into itself considering the identification $L(M_n, M_n) \cong M_{n^2}$. This construction makes a doorway to study quantum maps and semigroups of quantum maps from another perspective. We can take the basis decomposition of any quantum map $\alpha \in L(M_n, M_n)$ with respect to these NEB and try to characterise them in terms of the corresponding coefficients D_α of the decomposition. Analogous to the Choi and Jamiolkowski's result on channel-state duality we can characterise completely positive(CP) maps in terms of the matrix D_α. Furthermore, we can give a characterisation of semigroups of CP maps in term its generators which leads to another proof of Lindblad-Gorini- Kossakowski-Sudarshan's theorem on generator of CP semigroup. And finally we establish a characterisation of semigroup k-positive maps in terms of its generators.

Nice Error Basis and Weyl Operators

Definition: (Nicer Error Basis/NEB)

Let G be a group of order n^2. The set $\mathscr{E} = \{\pi_g \in U(n) : g \in G\}$ is called nice error basis if

i. $\pi_1 = Id_n$,

ii. $Tr(\pi_g) = \delta_{g,1}$,

iii. $\pi_g \pi_h = \omega(g,h)\pi_{gh}$ where $\omega : \mathbb{C} \times \mathbb{C} \longrightarrow \mathbb{C}$.

G is called the index group of the NEB.

A NEB forms an orthonormal basis (ONB) of $M_n(\mathbb{C})$ (up to scaling) with respect to the Hilbert-Schmidt inner product $\langle A, B\rangle := Tr(A^*B)$.

Example: Take the map $\xi(k,l) = \exp \dfrac{2\pi i k l}{n}$ on the abelian group $\mathbb{Z}_n \times \mathbb{Z}_n$. Define two unitary operators U_a and V_b for $a, b \in \mathbb{Z}_n$ on $\mathbb{C}^n$ by its action on the basis $\{\,|x\rangle : x \in \mathbb{Z}_n\}$

$$U_a|x\rangle := |x+a\rangle \qquad \text{and} \qquad V_b|x\rangle := \xi(b,x)|x\rangle$$

They satisfy the Weyl commutation relation $U_a V_b = \xi(a,b) V_b U_a$. If we define the discrete Weyl operators $W_{a,b}$ as product of these two i.e.

$$W_{a,b} := U_a V_b$$

Then $\{W_{a,b} : a, b \in \mathbb{Z}_n\}$ is a NEB of M_n.

Convenient Basis of $L(M_n, M_n)$

Proposition: Let G be an index group with NEB $\{\pi_g : g \in G\}$. We define a linear map $T_{x,y} : M_n(\mathbb{C}) \longrightarrow M_n(\mathbb{C})$ for $x, y \in G$ by

$$T_{x,y}(A) = \pi_x A \pi_y^* \qquad \text{for any } A \in M_n.$$

Then $\{1/n T_{x,y} : x, y \in G\}$ is an ONB of $L(M_n, M_n)$.

Therefore any $\alpha \in L(M_n, M_n)$ can be decomposed

$$\alpha(X) = \sum_{x,y\in G} D_\alpha(x,y) T_{x,y}(X) = \sum_{x,y\in G} D_\alpha(x,y)\pi_x X \pi_y^*.$$

Also we can express D_α as

$$D_\alpha(x,y) = 1/n^2 \sum_{g\in G} Tr(\pi_y \pi_g^* \pi_x^* \alpha(\pi_g)).$$

Characterisation of Quantum Maps

Theorem: Let $\{\pi_x\}_{x=1}^{n^2}$ be a basis of $M_n(\mathbb{C})$. Then a linear map $\alpha \in L(M_n, M_n)$ is positive if and only if $\forall u, v \in \mathbb{C}^n$

$$\langle u \otimes v \,|\, \tilde{\alpha}(u \otimes v)\rangle \geq 0,$$

Where $\tilde{\alpha} = \tau \circ \sum_{x,y} D_\alpha(x,y)\pi_x \otimes \pi_y^*$ and $\tau(u \otimes v) = v \otimes u$ is the flip operator.

Theorem: A linear map $\alpha \in L(M_n, M_n)$ is completely positive if and only if the corresponding matrix D_α is positive.

Proposition: Let G be an index group of a NEB. A linear map $\alpha \in L(M_n, M_n)$ is trace preserving if and only if

$$\sum_x \omega(x,g) D_\alpha(x, xg) = \delta_{g,1} \quad \text{for all } g \in G.$$

If $\{\pi_g : g \in G\}$ is a NEB then $M_n = span\{\pi_g : g \in G\}$. We have natural coalgebra structure on the dual M_n^* given by the comultiplication $\Delta_{M_n^*}$ and counit $\delta_{M_n^*}$

$$\Delta_{M_n^*}(\phi) = \phi \circ m \quad \text{and} \quad \delta(\phi) = \phi(Id_n).$$

$M_n \otimes \overline{M_n}$ inherits the natural coalgebra structure of M_n. For any two linear functional $\phi, \psi \in (M_n \otimes \overline{M_n})^*$ we define the convolution product

$$\phi \star \psi := (\phi \otimes \psi) \circ \Delta_{M_n \otimes \bar{M}_n}$$

where $\Delta_{M_n \otimes \bar{M}_n}$ is the comultiplication on $M_n \otimes \overline{M_n}$.

Consider the dual basis $\{1_x : x \in G\}$ of M_n i.e. $\langle \pi_x, 1_y\rangle = \delta_{x,y}$. We can identify D_α as a linear functional on the coalgebra $M_n \otimes \overline{M_n}$ via

$$D_\alpha(1_x \otimes \overline{1_y}) := D_\alpha(x,y)$$

Then we have the following isomorphism between $L(M_n, M_n)$ and $(M_n \otimes \overline{M_n})^*$

Proposition: $D : L(M_n, M_n) \ni \alpha \mapsto D_\alpha \in (M_n \otimes \overline{M_n})^*$ is an isomorphism i.e.

$$D_{\alpha \circ \beta} = D_\alpha \star D_\beta.$$

Semigroup of Quantum Maps

We can identify any linear functional ϕ on $M_n \otimes \overline{M_n}$ with a sesquilinear form on M_n by

$$\langle v, w\rangle_\phi := \phi(v \otimes \bar{w}) \qquad \text{for } v, w \in M_n.$$

So we can think of D_α as sesquilinear form on $M_n \otimes \overline{M_n}$.

A sesquilinear form K on a coalgebra (V, Δ, δ) is called **conditionally positive** if $K(v,v) \geq 0$ for all $v \in Ker(\delta)$. If C is a cone inside V then K is called **conditionally positive on the cone** C if $K(v,v) \geq 0$ for all $v \in C \cap Ker(\delta)$.

Theorem: Let $(\alpha_t)_{t\geq 0}$ be a semigroup of linear maps on $M_n(\mathbb{C})$. Then α_t is completely positive if and only if the sesquilinear form K with coefficient

$$K(x,y) = \frac{d}{dt}|_{t=0} D_{\alpha_t}(x,y)$$

is conditionally positive.

Theorem: Let $(\alpha_t)_{t\geq 0}$ be a semigroup of linear maps on $M_n(\mathbb{C})$. Then α_t is k-positive if and only if

$$K(x,y) = \frac{d}{dt}|_{t=0} D_{\alpha_t}(x,y)$$

is conditionally positive on the cone $D(\mathscr{P}_k)^+$, where $\mathscr{P}_k$ is the cone of k-positive maps.