The combinatorics of random tensors:
from random geometry to strongly-coupled phenomena

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Random Tensors at CIRM, Marseille
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Large-N expansion

[Gurau '11; Bonzom, Gurau, Rivasseau '12]

Scaling of bubbles and Feynman expansion governed by Gurau degree ω:

$$\mathcal{F}(\{\lambda_{\mathcal{B}}\}) = \ln \int \mathrm{d}T \, \exp\left(-\overline{T}\cdot T + \sum_{\mathcal{B}} \lambda_{\mathcal{B}}\, N^{-\frac{2}{(D-2)!}\omega(\mathcal{B})}\, \mathrm{Tr}_{\mathcal{B}}(\overline{T}, T)\right)$$
$$= \sum_{\omega\in\mathbb{N}} N^{D-\frac{2}{(D-1)!}\omega}\, \mathcal{F}_{\omega}(\{\lambda_{\mathcal{B}}\})$$

where $$\omega(\Delta) = D - n_{D-2}(\Delta) + \frac{D(D-1)}{4} n_D(\Delta)$$

- $\omega \in \mathbb{N}$
- generalization of the genus: $D = 2 \Rightarrow \omega = g$

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- generalization of the genus: $D = 2 \Rightarrow \omega = g$
- *not* a topological invariant of Δ when $D \geq 3$
- however: $\omega = 0 \Rightarrow \Delta$ is a D-sphere

Botanical interlude: melon diagrams

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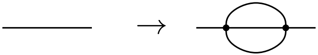

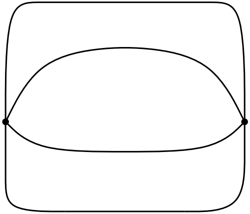

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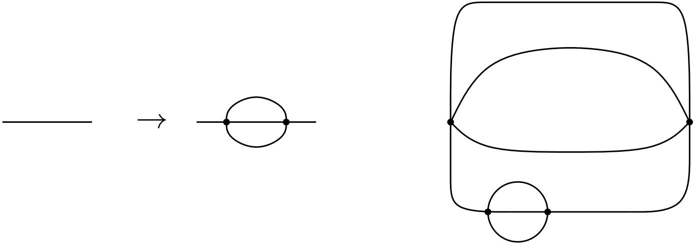

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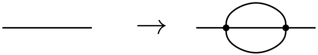

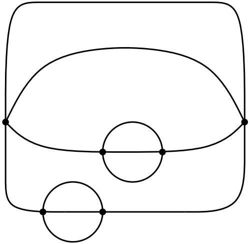

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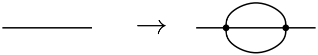

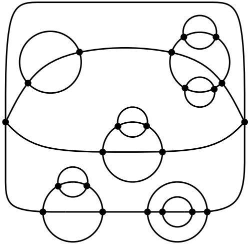

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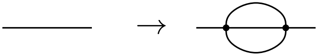
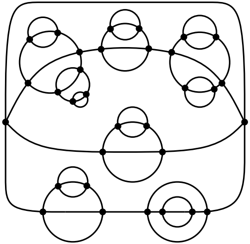
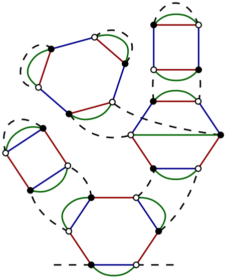




Leading order

[Bonzom, Gurau, Riello, Rivasseau '11;...]



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$\rightarrow$ special triangulations of the D-sphere, with a tree-like combinatorial structure.

Closed equation for their generating function:

$$\boxed{G(\lambda) = 1 + \lambda G(\lambda)^{D+1}} \qquad \text{(Fuss-Catalan)}$$

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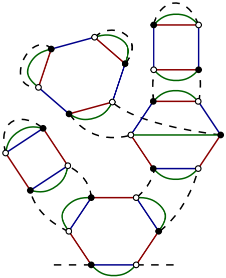

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Critical behaviour:

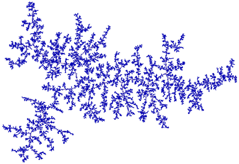
$$G(\lambda_c) - G(\lambda) \underset{\lambda \to \lambda_c}{\sim} K\,(\lambda_c - \lambda)^{1/2}$$

$$\Leftrightarrow \quad \#\{\text{rooted melonic } \Delta\} \sim K \lambda_c^{-n_\Delta} {n_\Delta}^{-3/2}$$

Universal critical exponent $3/2$ associated to combinatorial trees.

Continuum limit [Gurau, Ryan '13]

Melons are branched polymers
i.e. they converge to the continuous random tree [Aldous '91].



Credit: I. Kortchemski (https://igor-kortchemski.perso.math.cnrs.fr/images.html)

$$\#\{\text{rooted melonic } \Delta\} \sim K \lambda_c^{-n_\Delta} n_\Delta{}^{-3/2}$$

$$\boxed{d_{\text{spectral}} = 4/3} \quad ; \quad \text{distance scale} \sim n_\Delta{}^{1/2} \quad \text{and} \quad d_{\text{Hausdorff}} = 2$$

$\Rightarrow$ strong universality: limit independent of D!

Further results

- Combinatorial classification of graphs at order $\omega > 0$: "it's melons all the way down". [Gurau, Schaeffer '13]
- Double-scaling. [Bonzom, Gurau, Kaminski, Dartois, Oriti, Ryan, Tanasa '13 '14]
- Schwinger-Dyson eq. $\rightarrow$ analogue of loop equations. [Gurau '11]
- Non-perturbative treatment. [Gurau '14]
- ...
- Applications in Group Field Theory: [Boulatov, Ooguri, '92... Freidel, Gurau, Oriti '00s '10s...]

  Melonic behaviour $\Rightarrow$ rigorous renormalization theorems [Ben Geloun, Rivasseau '11; SC, Oriti, Rivasseau '13;...] [Review SC '16]

Beyond branched polymers?

No-go:

- Non-melonic large-N limits have been explored.
  [Bonzom, Delpouve, Rivasseau '15; Bonzom, Lionni '16; Lionni, Thüringen '17]
- Universality theorem: $D = 3 \Rightarrow$ branched polymers for arbitrary spherical bubbles. [Bonzom '18]

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Yes go?

- D even $\Rightarrow$ Brownian sphere, branched polymers and mixtures.
  [Bonzom, Delpouve, Rivasseau '15]
- Simple combinatorial restrictions may change the universality class:

$$\text{branched polymers} \underset{\text{2PI}}{\longrightarrow} \text{Ising on a random surface}$$

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Major open question:
genuinely new random geometric phase suitable for QG in $D \geq 3$?
[Lionni, Marckert '19]

Summary

Tensor models for random geometry:

- well-defined generalization of the matrix models approach;
- reproduce previously known universality classes: continuous random tree, Brownian sphere, and mixtures;
- tend to be dominated by tree-like combinatorial species $\Rightarrow$ no genuinely new universality class discovered so far...

  ...but a vast parameter space remains to be explored.

Entry points into the literature:

- "The Tensor Track" I-IV, Rivasseau, 2011-2016;
- "Random tensors", Gurau, 2016;
- "Colored Discrete Spaces", Lionni, 2018;
- "Combinatorial Physics", Tanasa, 2021.

Outline

Lecture 2

Colored O(N) models

The melonic limit as a window into strongly coupled physics

Irreducible random tensor ensembles

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Colored O(N) models

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Random tensors

Space of tensors $T = T_{a_1 \dots a_p}$, $a_i \in \{1, \dots, N\}$, equipped with measure of the form:

$$\mathrm{d}\nu(T) = \mathrm{d}\mu_{\boldsymbol{P}}(T) \mathrm{e}^{-S_N(T)}$$

- $\mathrm{d}\mu_{\boldsymbol{P}}$ is Gaussian with covariance $\boldsymbol{P}$:
- both $\boldsymbol{P}$ and S_N are invariant under the action of a unitary group: $\mathrm{O}(N)$, $\mathrm{U}(N)$ or $\mathrm{Sp}(N)$.

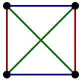
What type of universal behaviour can we obtain in the asymptotic limit $N \to \infty$?

Colored O(N) models

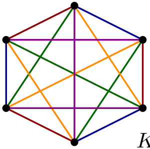
$T_{a_1 a_2 \dots a_p}$, in fundamental representation of O(N) $\times$ O(N) $\times \cdots \times$ O(N):

- $P_{a_1 a_2 \dots a_p, b_1 b_2 \dots b_p} = \delta_{a_1 b_1} \delta_{a_2, b_2} \cdots \delta_{a_p, b_p}$

- $S_N \propto$ complete-graph interaction



$K_4 \quad (p = 3)$



$K_6 \quad (p = 5)$

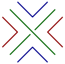
Theorem: (Ferrari, Rivasseau, Valette '17)
A melonic large N limit exists for prime $p \geq 3$.

$p = 3$: [SC, Tanasa '15]

Colored O(N) models

[SC, Tanasa '15]

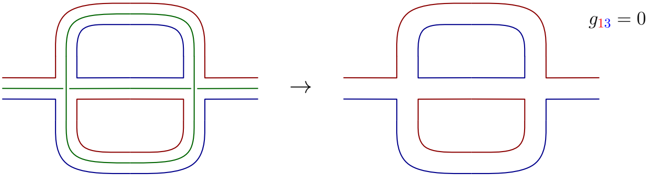
($p = 3$) $\frac{\lambda}{N^{3/2}} T_{aeb} T_{cfb} T_{ced} T_{afd}$



- $A(G) \sim N^{-\omega}$ with $\omega = 3 + \frac{3}{2}V - F \geq 0$
- G leading order $\Leftrightarrow$ $\omega = 0$ $\Leftrightarrow$ G is a melon diagram

Idea of proof:

- Euler relation: $\omega := g_{13} + g_{12} + g_{23} \in \frac{\mathbb{N}}{2}$, where g_{ij} = genus of a ribbon diagram.



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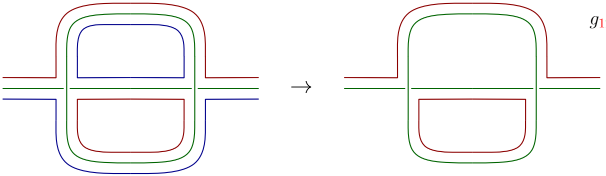

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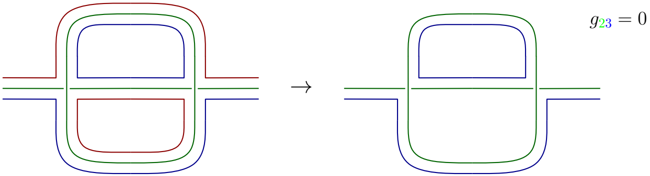

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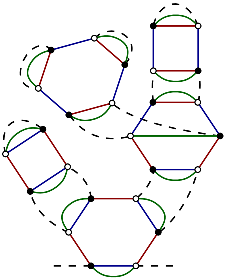

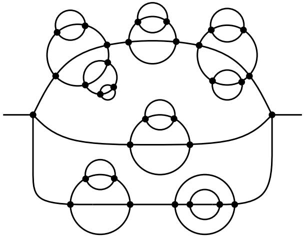

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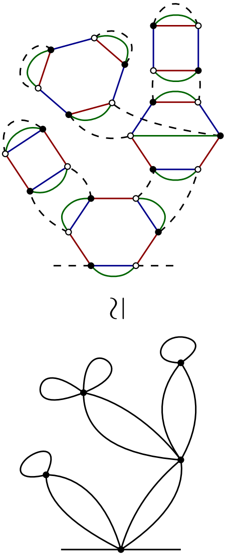

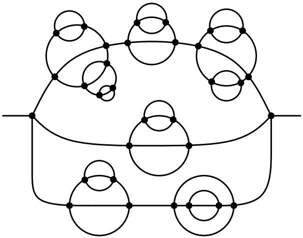

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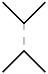
Irreducible random tensor ensembles

Three generic families of large N limits

Vector field $\phi_a(x)$

$\frac{\lambda}{N}(\phi_a\phi_a)^2$



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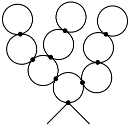

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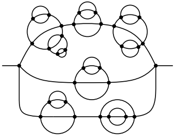
Tensor field $T_{abc}(x)$

$\frac{\lambda}{N^{3/2}}T_{aeb}T_{bfc}T_{ced}T_{dfa}$

Melon diagrams



Tractable

Matrix field $M_{ab}(x)$

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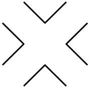

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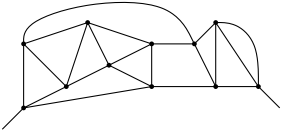

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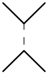

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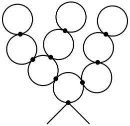

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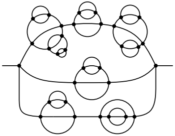

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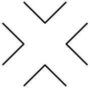

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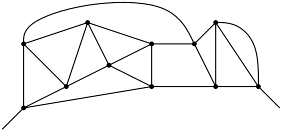

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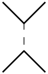

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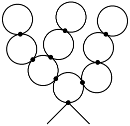

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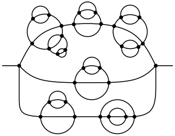

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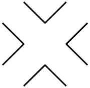

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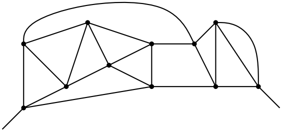

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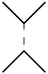

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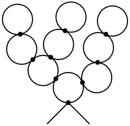

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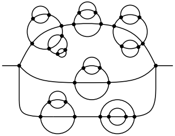
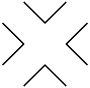


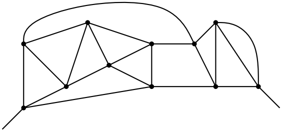
Tractable

Matrix field $M_{ab}(x)$

$\frac{\lambda}{N} M_{ab}M_{bc}M_{cd}M_{da}$



Planar diagrams



Hard

Melonic regime $\Rightarrow$ closed and often solvable systems of Schwinger-Dyson equations, capturing bilocal effects.

Sachdev-Ye-Kitaev model

[Sachdev, Ye, Georges, Parcollet '90s...; Kitaev '15, Maldacena, Stanford, Polchinski, Rosenhaus...]

- Disordered system of N Majorana fermions ψ_a in $d = 0 + 1$

$$H \sim J_{abcd}\, \psi_a\, \psi_b\, \psi_c\, \psi_d\,, \qquad \langle J_{abcd} \rangle = 0\,, \quad \left\langle J^2_{abcd} \right\rangle \sim \frac{\lambda^2}{N^3}$$

- Many interesting properties:
  - solvable at large N
  - emergent conformal symmetry at strong coupling
  - same effective dynamics as Jackiw-Teitelboim 2D quantum gravity $\rightarrow$ toy-models of quantum black holes
  - maximal quantum chaos [Maldacena, Shenker, Stanford]

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  - maximal quantum chaos [Maldacena, Shenker, Stanford]

- Same melonic large N limit as tensor models [Witten '16]

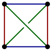
$\rightarrow$ SYK-like tensor quantum-mechanical models:

- same qualitative properties at large N and strong coupling;
- **no disorder**.

Klebanov-Tarnopolsky model [Klebanov, Tarnopolsky '16]

Tensor quantum mechanics of N^3 Majorana fermions:

$$S = \int dt \left(\frac{\mathrm{i}}{2} \psi_{i_1 i_2 i_3} \partial_t \psi_{i_1 i_2 i_3} + \frac{\lambda}{4N^{3/2}} \psi_{i_1 i_2 i_3} \psi_{i_4 i_5 i_3} \psi_{i_4 i_2 i_6} \psi_{i_1 i_5 i_6} \right)$$



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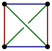
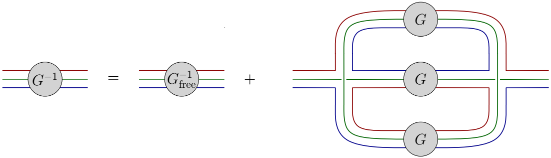


- Melonic dominance at large N $\Rightarrow$ closed Schwinger-Dyson equation: [SC, Tanasa '15]



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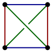

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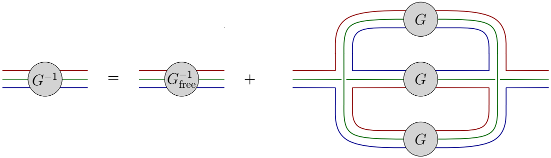


- SYK melonic equation: $\langle T(\psi_{a_1 a_2 a_3}(t_1) \psi_{b_1 b_2 b_3}(t_2)) \rangle \equiv G(t_1, t_2) \prod_{i=1}^{3} \delta_{a_i, b_i}$

$$G(t_1, t_2) = G_{\text{free}}(t_1, t_2) + \lambda^2 \int \mathrm{d}t\mathrm{d}t' \, G_{\text{free}}(t_1, t) \, [G(t, t')]^3 \, G(t', t_2)$$

STRONG-COUPLING REGIME

$$G(t_1, t_2) = G_{\text{free}}(t_1, t_2) + \lambda^2 \int \mathrm{d}t\mathrm{d}t'\, G_{\text{free}}(t_1, t)\,[G(t, t')]^3\, G(t', t_2)$$

- At strong coupling:

$$\lambda^2 \int \mathrm{d}t\, G(t_1, t)\,[G(t, t_2)]^3 = -\delta(t_1 - t_2)$$

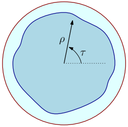
- Emergent conformal invariance: reparametrization $t \mapsto f(t)$

$$G(t_1, t_2) \mapsto |f'(t_1) f'(t_2)|^{1/4} G(f(t_1), f(t_2))$$

- Symmetry breaking: f governed by same dynamics as boundary modes in Jackiw-Teitelboim 2D quantum gravity

  $\Rightarrow$ "near AdS_2 / near CFT_1 correspondence"

[Kitaev '15; Maldacena, Stanford; Gross, Rosenhaus;...]



$ds^2 = d\rho^2 + \sinh^2\rho\, d\tau^2$

- solvable model of quantum black hole
- $\sim$ topological recursion for Weil-Petersson volumes

[Saad, Shenker, Stanford '19; Mirzakhani '07; Eynard-Orantin '07]

Tensor field theory

Unlike SYK, tensor models naturally fit in the framework of local quantum field theory.

QFT generalization

Rely on tensor models to construct melonic theories in $d > 1$.

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QFT generalization

Rely on tensor models to construct melonic theories in $d > 1$.

Why it is interesting:

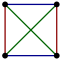
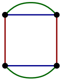
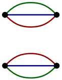
- only diagrams that proliferate are melons and ladder diagrams
  $\Rightarrow$ explicit non-perturbative resummation sometimes possible
- melons are bi-local
  $\Rightarrow$ anomalous dimensions $\Rightarrow$ non-trivial CFTs and RG flows
- 4-point functions = sums of ladder diagrams
  $\Rightarrow$ non-perturbative access to the spectrum

$\Rightarrow$ mathematically precise insights into strongly-coupled QFT.

Long-range bosonic models [Benedetti, Gurau, Harribey '19]

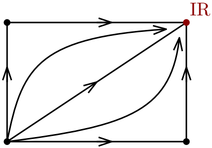
Bosonic tensor field theory in $d < 4$: $\zeta = \frac{d}{4}$

$$\mathcal{L} = \frac{1}{2}\varphi_{abc}(-\Delta)^{\zeta}\varphi_{abc} + \frac{m^{2\zeta}}{2}\varphi_{abc}\varphi_{abc}$$

$$+ \frac{\mathrm{i}\,\lambda}{4N^{3/2}}$$  $$+ \frac{\lambda_P}{4N^2}$$  $$+ \frac{\lambda_D}{4N^3}$$ 

- Large-N melonic limit $\Rightarrow$ *explicit* renormalization group flow to a unitary CFT in the IR:

  [Benedetti, Gurau, Harribey, Suzuki '19; Benedetti, Gurau, Suzuki '20]



- Allows to test paradigms of QFT in rigorous set-ups

  e.g. validity of F-theorem [Benedetti, Gurau, Harribey, Lettera '21]

Outline

Colored O(N) models

The melonic limit as a window into strongly coupled physics

Irreducible random tensor ensembles

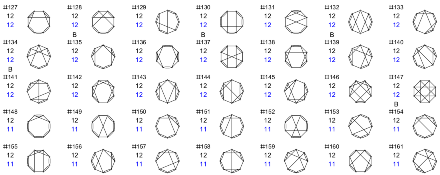
Generic tensors

Conjecture (Klebanov–Tarnopolsky '17)

For $p = 3$, $\exists$ melonic large N limit for $O(N)$ symmetric **traceless** tensors.

Evidence. Explicit numerical check of all diagrams up to order λ^8.

[Klebanov, Tarnopolsky, JHEP '17]



Proof and further generalizations.

1. $O(N)$ irreducible, $p = 3$

   [Benedetti, SC, Gurau, Kolanowski, Commun. Math. Phys. '19; SC, JHEP '18]
2. $Sp(N)$ irreducible, $p = 3$ [SC, Pozsgay, Nucl. Phys. B '19]
3. $O(N)$ irreducible, $p = 5$ [SC, Harribey, Commun. Math. Phys. '22]

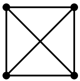
Much more involved and subtle constructions than in the colored case.

O(N) IRREDUCIBLE MODELS

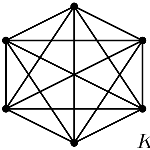
Real p-index tensor $T_{a_1 \dots a_p}$, with p odd and measure of the form:

$$d\nu(T) = d\mu_{\boldsymbol{P}}(T) e^{-S_N(T)}$$

- $\boldsymbol{P}$ = orthogonal projector on an irreducible representation of O(N);
- $S_N = -\frac{\lambda}{N^\alpha} \mathrm{Inv}(T)$, where $\mathrm{Inv}(T)$ is a complete-graph invariant (graph K_{p+1}).



$K_4 \quad (p = 3)$



$K_6 \quad (p = 5)$

Do these models admit large N expansions? Are they melonic?

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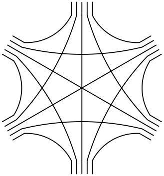
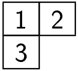
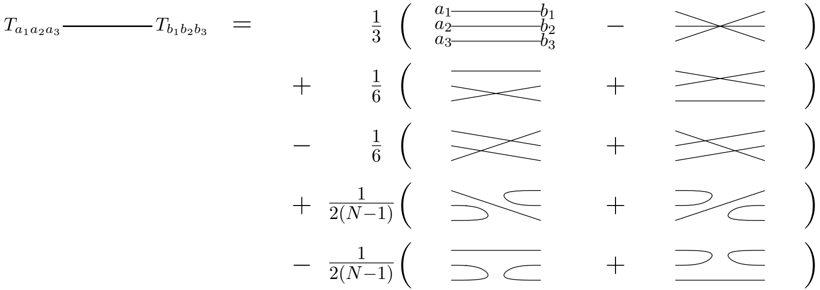


Do these models admit large N expansions? Are they melonic?

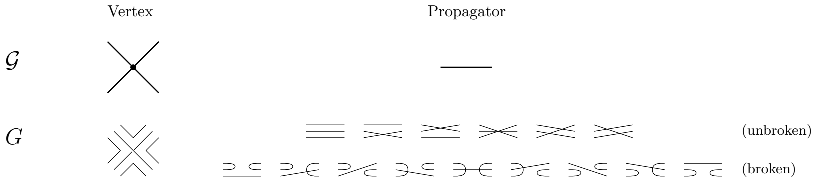
P = orthogonal projector on one of the irreducible tensor spaces.

<u>example:</u> for traceless tensors with symmetry





Irreducible tensors – Feynman amplitudes



$$\mathcal{F}_N(\lambda) = \sum_{\text{connected maps } \mathcal{G}} \frac{\lambda^{V(\mathcal{G})}}{s(\mathcal{G})} A(\mathcal{G})$$

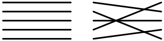
$\mathcal{G}$ decomposes into up to $15^{E(\mathcal{G})}$ stranded graphs G:

$$A(\mathcal{G}) = \sum_G A(G)\,, \qquad A(G) \sim N^{-\omega(G)}$$

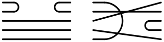
$$\omega(G) = 3 + \frac{3}{2}V(G) + B(G) - F(G)$$

$V = \#\{\text{vertices}\}$, $B = \#\{\text{broken edges}\}$, $F = \#\{\text{faces}\}$

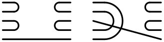
Irreducible tensors – 5-index tensors



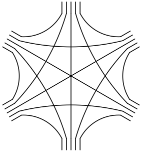
Unbroken



Broken



Doubly-broken



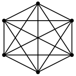
Map $\mathcal{G}$ decomposes into up to $945^{E(\mathcal{G})}$ stranded graphs G:

$$A(\mathcal{G}) = \sum_{G} A(G)\,, \qquad A(G) \sim N^{-\omega(G)}$$

$$\omega(G) = 5 + 5V(G) + B_1(G) + 2B_2(G) - F(G)$$

$B_1 = \#\{\text{broken edges}\}$, $B_2 = \#\{\text{doubly} - \text{broken edges}\}$

MAIN THEOREMS $(p = 5)$

$$Z_{\boldsymbol{P}}(\lambda, N) = \int \mathrm{d}\mu_{\boldsymbol{P}} \exp\left(\frac{\lambda}{6N^5}$$  $$\right) \qquad F_{\boldsymbol{P}}(\lambda, N) = \frac{6}{N^5} \lambda \partial_\lambda \ln Z_{\boldsymbol{P}}(\lambda, N)$$

Theorem 1 (SC, Harribey '21) In the sense of formal power series:

$$F_{\boldsymbol{P}}(\lambda, N) = \sum_{\omega \in \mathbb{N}} N^{-\omega} F_{\boldsymbol{P}}^{(\omega)}(\lambda)$$

Theorem 2 (SC, Harribey '21) For sufficiently small λ, $F_{\boldsymbol{P}}^{(0)}(\lambda)$ is the unique continuous solution of the polynomial equation

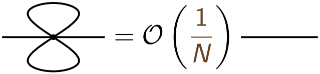
$$1 - X + m_{\boldsymbol{P}} \lambda^2 X^6 = 0$$

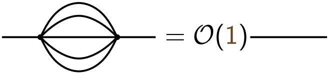
such that $F_{\boldsymbol{P}}^{(0)}(0) = 1$, and where $m_{\boldsymbol{P}}$ is a model-specific real constant.

Example. For the symmetric traceless and antisymmetric reps, $m_{\boldsymbol{P}} = \left(\frac{1}{5!}\right)^4$.

Proof strategy

1. Eliminate melon and double-tadpole 2-point functions at the Feynman map level:





This is where the irreducibility assumption plays a crucial role.

2. Obtain $\mathcal{G}$ with **no** melon and **no** double-tadpole.

<u>Proposition:</u> For any stranded configuration G of $\mathcal{G}$, $\omega(G) \geq 0$.

Proof. Induction on $V = \#\{\text{vertices}\}$. Conceptually straightforward but challenging by its complexity.

Idea of proof – combinatorial moves

Find local combinatorial moves that:

- decrease V;
- decrease ω;
- preserve constraints: connectedness, $\emptyset$ melon, $\emptyset$ double-tadpole.

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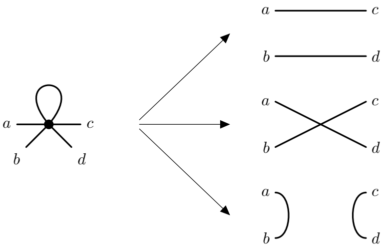

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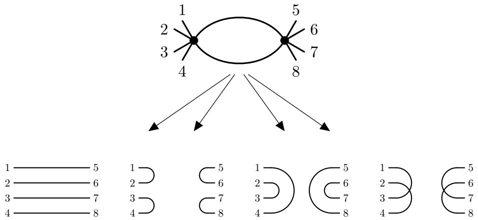

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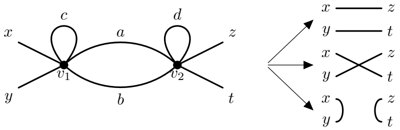

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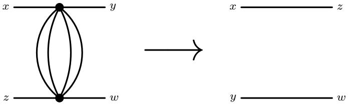

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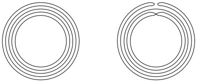
End graphs

- Ring graphs ($V = 0$):  $\cdots$
- G with no face of length 1 or 2 $\Rightarrow \omega(G) > 0$.
- Special cases that need to be treated separately.

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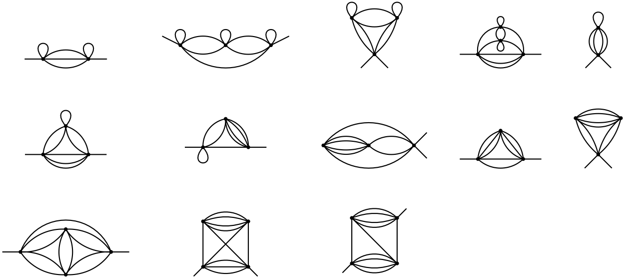
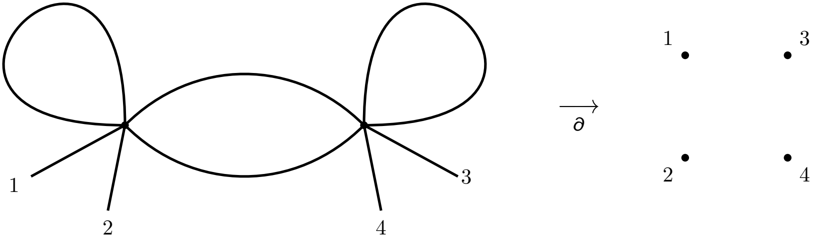


Idea of proof – boundary graphs



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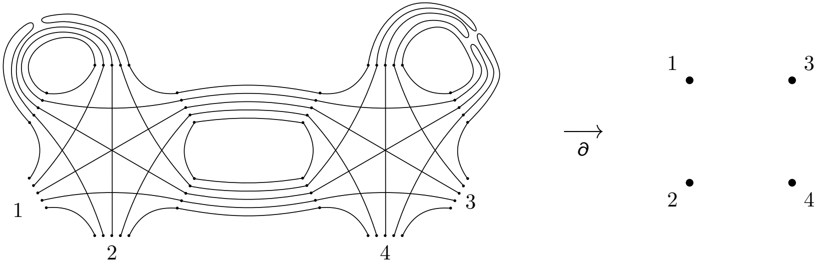

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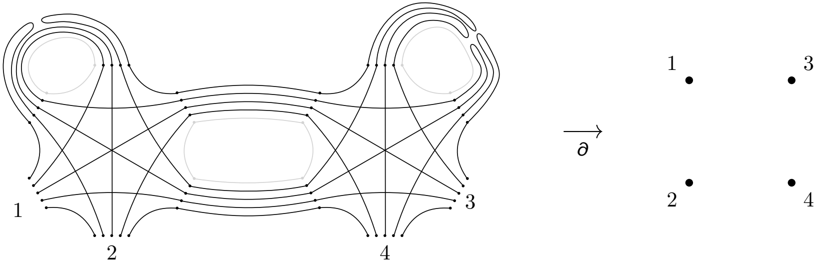

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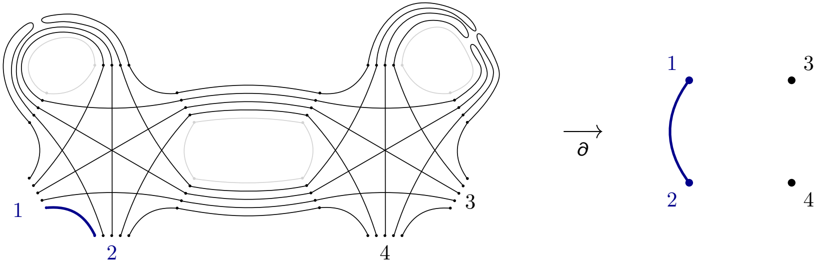

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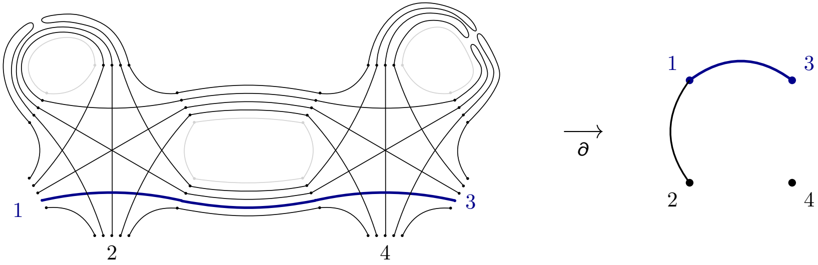

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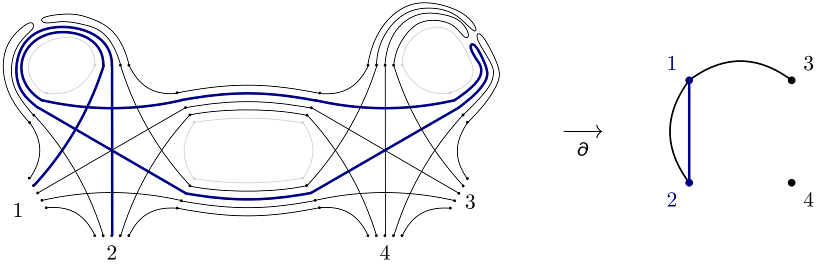
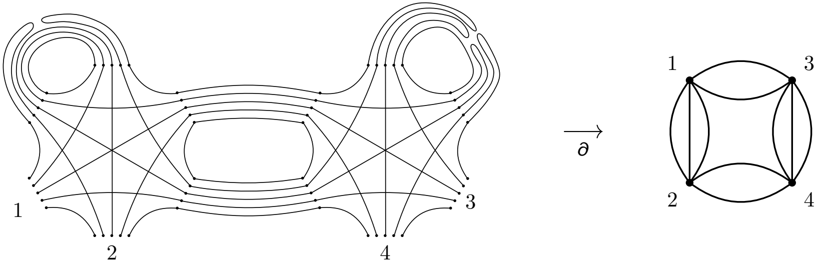
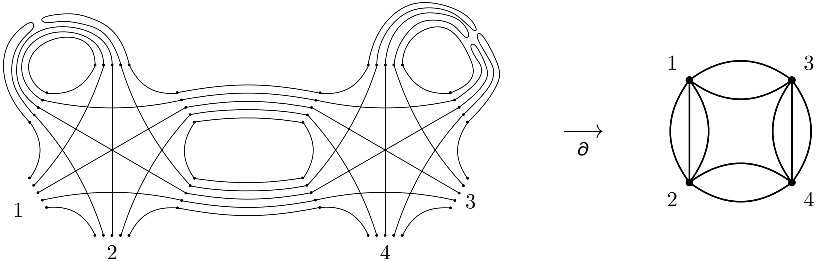
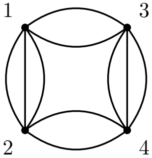


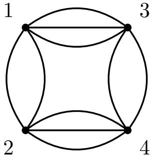
Idea of proof – boundary graphs



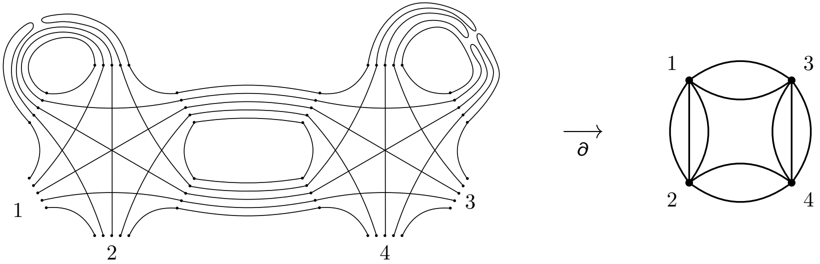


One can recast recursive bounds on ω into bounds on **flip distance** between boundary graphs:





IDEA OF PROOF – BOUNDARY GRAPHS



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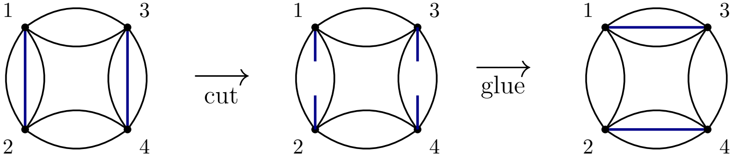
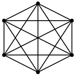


Main theorems

$$Z_{\boldsymbol{P}}(\lambda, N) = \int d\mu_{\boldsymbol{P}} \exp\left(\frac{\lambda}{6N^5}\right.$$  $$\left.\right) \qquad F_{\boldsymbol{P}}(\lambda, N) = \frac{6}{N^5}\lambda\partial_\lambda \ln Z_{\boldsymbol{P}}(\lambda, N)$$

Theorem 1 (SC, Harribey '21) In the sense of formal power series:

$$\checkmark \qquad F_{\boldsymbol{P}}(\lambda, N) = \sum_{\omega \in \mathbb{N}} N^{-\omega} F_{\boldsymbol{P}}^{(\omega)}(\lambda)$$

Theorem 2 (SC, Harribey '21) For sufficiently small λ, $F_{\boldsymbol{P}}^{(0)}(\lambda)$ is the unique continuous solution of the polynomial equation

$$1 - X + m_{\boldsymbol{P}}\lambda^2 X^6 = 0$$

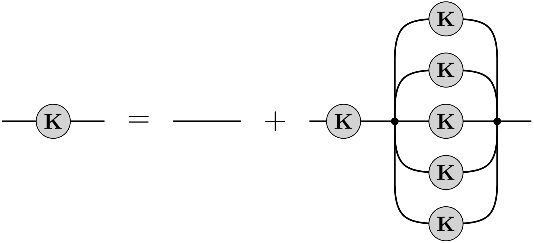
such that $F_{\boldsymbol{P}}^{(0)}(0) = 1$, and where $m_{\boldsymbol{P}}$ is a model-specific real constant.

Example. For the symmetric traceless and antisymmetric reps, $m_{\boldsymbol{P}} = \left(\frac{1}{5!}\right)^4$.

Melonic dominance

Proposition: $\mathcal{G}$ is leading order $\Leftrightarrow$ $\mathcal{G}$ is melonic.

Hallmark of melonic limit: the 2-point function verifies a closed SDE



$\Rightarrow F_{\boldsymbol{P}}^{(0)}$ is a solution of the polynomial equation

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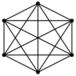

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Summary

Tensor models for srongly-coupled quantum theory:

- melonic limit exended from colored to generic tensor ensembles;
- provides third generic family of large N theories, both rich and tractable;
- can reproduce SYK-like physics without disorder;
- generalize to QFT $\rightarrow$ new family of large N QFTs which can be studied analytically.

Entry points into the literature:

- "TASI Lectures on Large N Tensor Models", Klebanov, Popov, Tarnopolsky, 2018;
- "The Tensor Track" V-VI, Rivasseau, Delporte, 2018-2020;
- "Notes on Tensor Models and Tensor Field Theories", Gurau, 2019;
- "Melonic CFTs", Benedetti, 2020.