The combinatorics of random tensors:
from random geometry to strongly-coupled phenomena

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Random Tensors at CIRM, Marseille
March 14-18, 2022

Random tensors

Space of tensors $T = T_{a_1 \dots a_p}$, $a_i \in \{1, \dots, N\}$, equipped with measure of the form:

$$\mathrm{d}\nu(T) = \mathrm{d}\mu_{\boldsymbol{P}}(T)\mathrm{e}^{-S_N(T)}$$

- $\mathrm{d}\mu_{\boldsymbol{P}}$ is Gaussian with covariance $\boldsymbol{P}$:

$$\int \mathrm{d}\mu_{\boldsymbol{P}}(T) T_{a_1 \dots a_p} T_{b_1 \dots b_p} = \boldsymbol{P}_{a_1 \dots a_p, b_1 \dots b_p}$$

- both $\boldsymbol{P}$ and S_N are invariant under the action of a product of unitary groups: $\mathrm{O}(N)$, $\mathrm{U}(N)$ or $\mathrm{Sp}(N)$.

What type of universal behaviour can we obtain in the asymptotic limit $N \to \infty$?

Large N expansion: basic idea

$$\mathcal{F}(\lambda, N) = \ln\left(\int d\mu_{\boldsymbol{P}}(T) e^{-\frac{\lambda}{N^{\alpha}} \mathrm{Inv}(T)} \right)$$

Main steps:

1. Formal perturbative expansion in λ.

   $\Rightarrow$ combinatorial interpretation: sum over Feynman graphs

2. Find α such that a $1/N$ expansion exists.

3. Resum $\mathcal{F}_{\omega}(\lambda)$.

4. Non-perturbative analysis of $1/N$ expansion.
   (won't be discussed in these lectures)

Large N expansion: basic idea

$$\mathcal{F}(\lambda, N) = \ln\left(\int \mathrm{d}\mu_{\boldsymbol{P}}(T)\mathrm{e}^{-\frac{\lambda}{N^\alpha}\mathrm{Inv}(T)}\right)$$
$$= \sum_{\text{graph } G} \frac{(-\lambda)^{V(G)}}{\mathrm{sym}(G)}\mathcal{A}(G)$$

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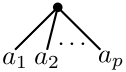
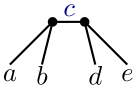
Large N expansion: main applications

- **Matrix models**
  - Random surfaces / 2D quantum gravity from matrix integrals.
  - Large N limit as an approximation tool in quantum (field) theory.
- **Tensor models**
  - Random geometry / quantum gravity in $D \geq 3$.
  - New generic class of large N theories: more solvable than matrix theories, but still physically interesting.

Tensors and invariants

Real symmetric tensor:

$T_{a_1 a_2 \cdots a_p} =$ 

$\sum_{c=1}^{N} T_{abc} T_{cde} =$ 

Connected invariants:

Tensors and invariants

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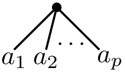
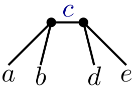
$T_{a_1 a_2 \cdots a_p} =$ 

$\sum_{c=1}^{N} T_{abc} T_{cde} =$ 

Connected invariants:

$p = 1$  $(\phi_a \phi^a)$

Tensors and invariants

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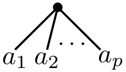
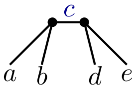
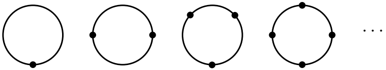
$T_{a_1 a_2 \cdots a_p} =$ 

$\sum_{c=1}^{N} T_{abc} T_{cde} =$ 

Connected invariants:

$p = 2$  $(\mathrm{tr}(M^n))$

Tensors and invariants

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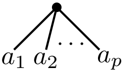
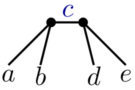
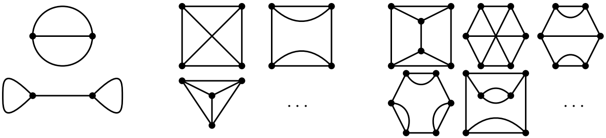
$T_{a_1 a_2 \cdots a_p} =$ 

$\sum_{c=1}^{N} T_{abc} T_{cde} =$ 

Connected invariants:

$p = 3$



$$\#\{\text{invariants of order } 2n\} \sim \left(\tfrac{3}{2}\right)^n n!$$

$\Rightarrow$ Rapid growth of theory space for $p \geq 3$:

- large N behaviour explicitly depends on the combinatorial structure of the invariants which contribute to the action;
- this dependence is hard to characterize in full generality.

Outline

Lecture 1

Large N expansion of matrix models

First generalization: complex colored tensor models

Random geometry applications

Lecture 2

Other ensembles of random tensors and QFT applications

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Large N expansion of matrix models

First generalization: complex colored tensor models

Random geometry applications

Hermitian matrix ensemble

$$\mathcal{Z}_N(\lambda) = \int_{H_N} \mathrm{d}M \exp\left(-N\left(\frac{1}{2}\mathrm{Tr}M^2 + \frac{\lambda}{4}\mathrm{Tr}M^4 + \dots\right)\right)$$

$$(\mathrm{d}M := \textstyle\prod_k \mathrm{d}M_{kk} \prod_{i<j} \mathrm{dRe}M_{ij}\,\mathrm{dIm}M_{ij})$$

- Basic question: determine expectation values of $\mathrm{U}(N)$-invariant observables
$$\langle \mathrm{Tr}(M^{n_1})\mathrm{Tr}(M^{n_2})\dots\mathrm{Tr}(M^{n_k})\rangle$$

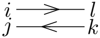
- Gaussian theory ($\lambda = 0$): entirely determined by the propagator

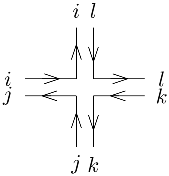
$$\boldsymbol{P}_{ij,kl} := \langle M_{ij}M_{kl}\rangle_0 = \frac{1}{\mathcal{Z}_N(0)}\int \mathrm{d}M\,\mathrm{e}^{-\frac{N}{2}\mathrm{Tr}M^2}\,M_{ij}M_{kl} = \frac{1}{N}\delta_{il}\delta_{jk}$$

Higher order moments computed by Wick's theorem.

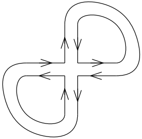
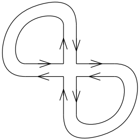
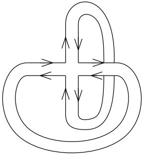
Gaussian correlators

Graphical representation of

- propagator: $\boldsymbol{P}_{ij,kl} = \frac{1}{N}\delta_{il}\delta_{jk} =$ 

- interaction: $N\mathrm{Tr}M^4 = N\sum_{i,j,k,l} M_{ij}M_{jk}M_{kl}M_{li} =$ 

Invariant correlators $\rightarrow$ ribbon diagrams

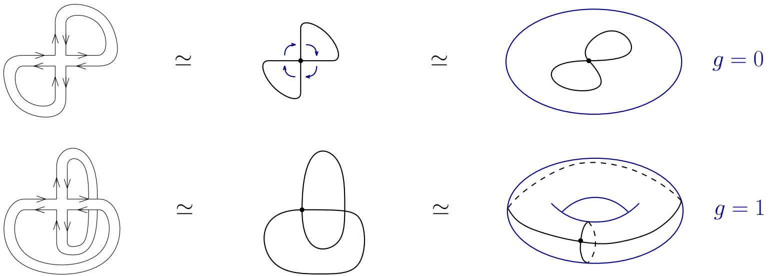
$\langle N\mathrm{Tr}M^4\rangle_0 \underset{\text{Wick}}{=}$  $+$  $+$ 

$$= N\left(\frac{1}{N}\right)^2\left(\quad N^3 \quad + \quad N^3 \quad + \quad N \quad\right)$$

$$= 2N^2 + 1$$

Ribbon diagrams

ribbon graph $\simeq$ combinatorial map $\simeq$ embedded graph



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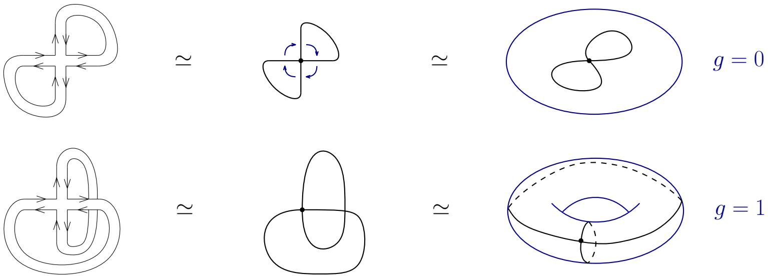


The weight / amplitude of an arbitrary ribbon graph only depends on the topology of the surface it represents:

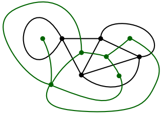
$$\boxed{N^{V-E+F} = N^{\chi} = N^{2c-2g}}$$

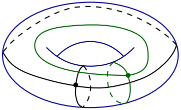
$V = \#\{\text{vertices}\}\,,\ E = \#\{\text{edges}\}\,,\ F = \#\{\text{faces}\}\,.$
$g = \text{genus},\ c = \#\{\text{connected components}\}.$

Topological expansion of matrix models ['t Hooft '74]

$$\mathcal{Z}_N(\lambda) = \int \mathrm{d}M \, \exp\left(-N\left(\frac{1}{2}\mathrm{tr}(M^2) + \frac{\lambda}{4}\mathrm{tr}(M^4)\right)\right)$$

$$= \sum_{\text{ribbon graph } G} \frac{(-\lambda)^{V(G)}}{s(G)} N^{\chi(G)} = \sum_{\text{quandrangulation } \Delta} \frac{(-\lambda)^{n(\Delta)}}{s(\Delta)} N^{\chi(\Delta)}$$





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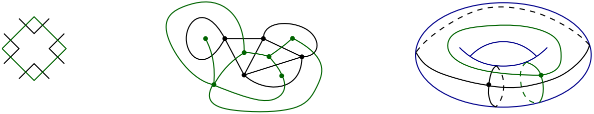
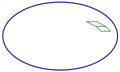
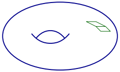
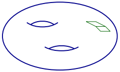
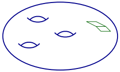


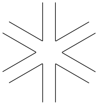
Universal large-N expansion

$$\ln \mathcal{Z}_N(\lambda) = \sum_{g \in \mathbb{N}} N^{2-2g} \, \mathcal{F}_g(\lambda) \qquad \text{with} \qquad \mathcal{F}_g(\lambda) = \sum_{\substack{G \text{ connected} \\ g(G)=g}} \frac{(-\lambda)^{V(G)}}{s(G)}$$

N^2  $+ N^0$  $+ N^{-2}$  $+ N^{-4}$  $+ \cdots$

Generalizations

- General potential: $\mathrm{Tr}(M^4) \to \mathrm{Tr}(V(M))$



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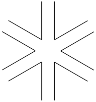
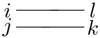


$\cdots$

- β-ensembles:

$\boldsymbol{P}_{ij,kl} \propto \frac{1}{N} \Big($  $- (1 - \frac{2}{\beta})$  $\Big)$

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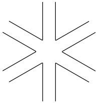

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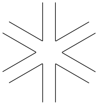

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$$\boldsymbol{P}_{ij,kl} \propto \frac{1}{N}\left(\begin{smallmatrix} i \\ j \end{smallmatrix} \overline{\underline{\qquad\quad}} \begin{smallmatrix} l \\ k \end{smallmatrix} - (1 - \frac{2}{\beta}) \begin{smallmatrix} i \\ j \end{smallmatrix} \asymp \begin{smallmatrix} l \\ k \end{smallmatrix} \right)$$

  - Hermitian models with U(N) symmetry ($\beta = 2$)
  - Real symmetric matrix with O(N) symmetry ($\beta = 1$).

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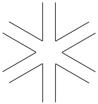

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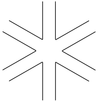

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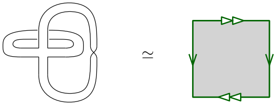
$\Rightarrow$ generate non-orientable surfaces.



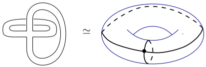
[Review: Eynard, Kimura, Ribault '15]

Applications of the large N limit

- Random surfaces and QG in $D = 2$

  Matrix integral at large $N \to$ statistical sum of

  Feynman graphs $\simeq$ Euclidean space-time geometries



- Strongly-coupled QFT

  Large number of fields/symmetries e.g. $\mathrm{SU}(3) \to \mathrm{SU}(N)$

  - perturbation theory in $1/N$
  - non-perturbative effects in coupling constants λ

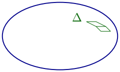
  Key probe of holographic dualities:

  - gauge theory $\leftrightarrow$ Einstein gravity
  - vector models $\leftrightarrow$ higher-spin gravity

What are tensor models good for in these two lines of thoughts?

QG in $D = 2$ as a matrix integral

$$\boxed{\ln \int \mathrm{d}M\, e^{-N\left(\frac{1}{2}\mathrm{tr}M^2 - \frac{\lambda}{q}\mathrm{tr}M^q\right)} \underset{N\to\infty}{\longrightarrow} \mathcal{F}_0(\lambda) = \sum_{\Delta} \lambda^{n_\Delta}}$$



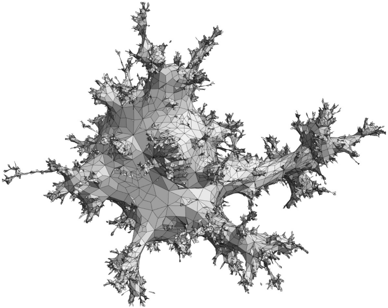
- Large-N limit $\Rightarrow$ generating function of planar q-angulations Δ, weighted by $n_\Delta \sim$ area.
- Critical regime: $\lambda \to \lambda_c \Rightarrow$ continuum limit.
- Double-scaling $\Rightarrow$ non-trivial sum over topologies.

Universality: the distribution over 2d metrics converges to the Brownian sphere in the continuum limit, independently of the details of the potential (e.g. value of q).

$\to$ basic random geometry behind Liouville QG.

Brownian sphere

[Le Gall, Miermont '13]



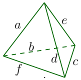
Credit: T. Budd (https://hef.ru.nl/~tbudd/gallery/)

$$\#\{\text{ rooted planar } \Delta\,\} \sim K \lambda_c^{-n_\Delta} {n_\Delta}^{-5/2}$$

$d_{\text{spectral}} = 2$; distance scale $\sim {n_\Delta}^{1/4}$ and $d_{\text{Hausdorff}} = 4$

QG in $D \geq 3$ as a D-index tensor integral?

$$\mathcal{F}(\lambda) = \ln \int \mathrm{d}T \, \exp\left(-T_{abc}T_{abc} + \frac{\lambda}{N^{\alpha}} T_{aeb}T_{bfc}T_{ced}T_{dfa} \right)$$



[Ambjørn, Durhuss, Jónsson '91; Gross '91; Sasakura '91;...]

- Challenges:
  - interplay between combinatorics and topology: nice global properties from local Feynman rules?
  - large-N expansion?
  - matrix techniques not available (spectral representation?)

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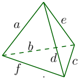

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- Path to progress: [Gurau '09; Gurau, Rivasseau, Bonzom,... '10s]
  - more symmetry: $\mathrm{U}(N)^D \rightarrow$ colored tensor models
  - tractable combinatorics, mapping to sufficiently regular topological spaces.

$\Rightarrow$ universal large-N expansion, in any $D \geq 3$

indexed by *Gurau degree* $\omega \geq 0$

Outline

Large N expansion of matrix models

First generalization: complex colored tensor models

Random geometry applications

Colored tensor models [Gurau '09]

Multipartite pure quantum state

$$|\Psi\rangle = \sum_{a_1,a_2,\dots,a_D} T_{a_1 a_2 \dots a_D} |a_1\rangle \otimes |a_2\rangle \otimes \cdots \otimes |a_D\rangle$$

with $a_k \in \{1, \dots, N_k\}$.

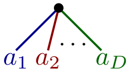
- Entanglement structure characterized by local unitary (LU) invariants:

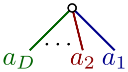
$$\mathrm{U}(N_1) \times \mathrm{U}(N_2) \times \cdots \times \mathrm{U}(N_D)$$

- LU invariant (and normalized) random $T_{a_1 a_2 \dots a_D}$
  $\sim$ distribution over multipartite pure state entanglement structures.

In the rest of the talk, take $N_k = N \gg 1$.

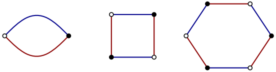
Colored tensor models [Gurau '09]

$T_{a_1 a_2 \cdots a_D} =$ 



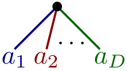
$= \overline{T}_{a_1 a_2 \cdots a_D}$

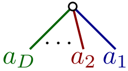
$\mathrm{U}(N)^D$ invariants indexed by bubble diagrams $\mathcal{B}$:

$(D = 2)$  $\cdots$

Colored tensor models

[Gurau '09]

$T_{a_1 a_2 \cdots a_D} =$ 



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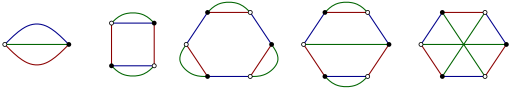
$(D = 3)$  $\cdots$

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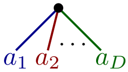
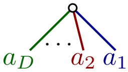
$T_{a_1 a_2 \cdots a_D} =$ 



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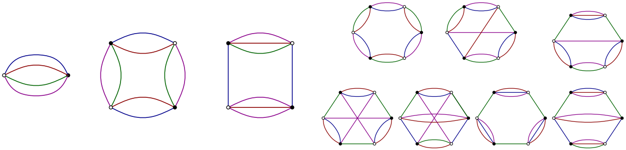
$(D = 4)$  $\cdots$

Colored tensor models

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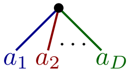
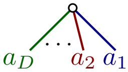
$T_{a_1 a_2 \cdots a_D} =$ 



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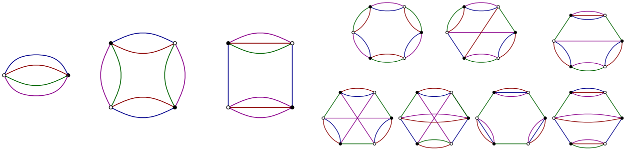
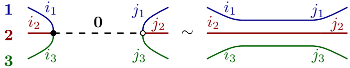
$(D = 4)$  $\cdots$

Partition function:

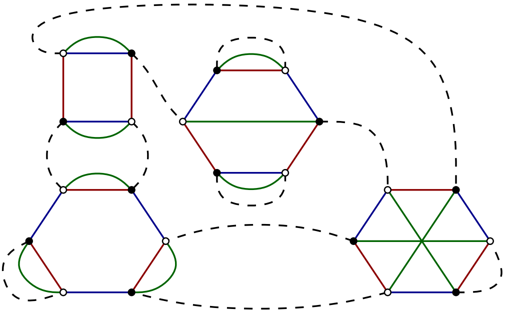
$$\mathcal{F}(\{\lambda_{\mathcal{B}}\}) = \ln \int \mathrm{d}T \, \exp\left(-\overline{T} \cdot T + \sum_{\mathcal{B}} \frac{\lambda_{\mathcal{B}}}{N^{\alpha(\mathcal{B})}} \mathrm{Tr}_{\mathcal{B}}(\overline{T}, T) \right)$$

FEYNMAN GRAPHS

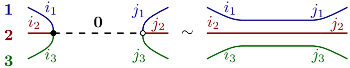


$= \delta_{i_1 j_1}\delta_{i_2 j_2}\delta_{i_3 j_3}$

G



Feynman graphs



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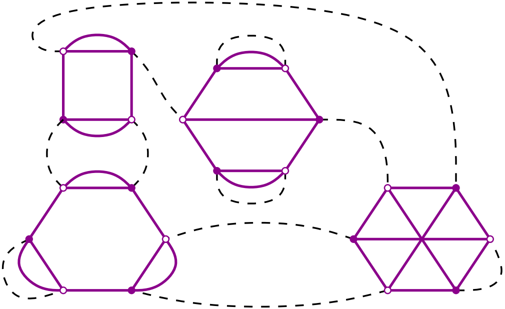

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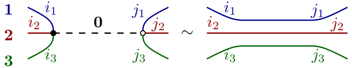

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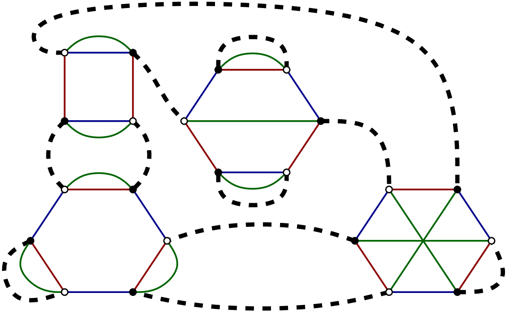
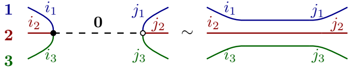


$V = \#\{\text{vertices}\}$; $p = \#\{\text{propagators}\}$

Feynman graphs



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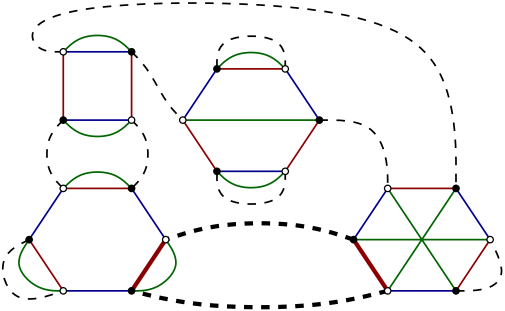

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$$\boxed{\mathcal{A}(G) \propto N^{\sum_j F_{\mathbf{0j}}}}$$

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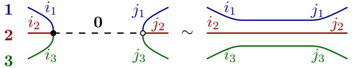

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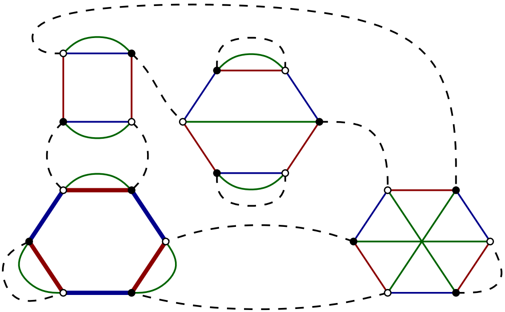

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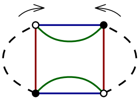
$$\boxed{\mathcal{A}(G) \propto N^{\sum_j F_{\mathbf{0j}}}}$$

Jackets

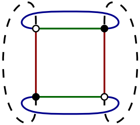
Colored graph + cyclic permutation σ on the colors

$\Rightarrow$ combinatorial map J_σ, called *jacket*.



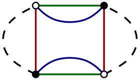
$\sigma = (\mathbf{0123})$

$g(J_\sigma) = 0$



$\sigma' = (\mathbf{0231})$

$g(J_{\sigma'}) = 1$



$\sigma'' = (\mathbf{0213})$

$g(J_{\sigma''}) = 0$

$J_\sigma \sim J_{\sigma^{-1}} \Rightarrow \exists\ \frac{D!}{2}$ inequivalent choices of σ.

E.g. 3 inequivalent jackets for $D = 3$.

Gurau degree

Definition *Gurau degree* of a $(D+1)$-colored graph G:

$$\omega(G) = D - F(G) + \frac{D(D-1)}{2} p(G)$$

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$$\forall G, \qquad \omega(G) \in \mathbb{N}$$

Proof.

$$\omega(G) = \frac{1}{(D-1)!} \sum_{\sigma} g(J_\sigma)$$

$\square$

Large-N expansion

[Gurau '11; Bonzom, Gurau, Rivasseau '12]

Scaling of bubbles and Feynman expansion governed by Gurau degree ω:

$$\mathcal{F}(\{\lambda_{\mathcal{B}}\}) = \ln \int \mathrm{d}T \, \exp\left(-\overline{T} \cdot T + \sum_{\mathcal{B}} \lambda_{\mathcal{B}} \, N^{-\frac{2}{(D-2)!}\omega(\mathcal{B})} \, \mathrm{Tr}_{\mathcal{B}}(\overline{T}, T) \right)$$
$$= \sum_{\omega \in \mathbb{N}} N^{D - \frac{2}{(D-1)!}\omega} \, \mathcal{F}_{\omega}(\{\lambda_{\mathcal{B}}\})$$

where $$\boxed{\omega(G) = D - F(G) + \frac{D(D-1)}{2} p(G)}$$

Generalization of the matrix genus expansion:

- $\omega \in \mathbb{N}$
- $D = 2 \Rightarrow \omega = g$

Topological/geometric interpretation?

Typical questions

- Combinatorial structure of leading-order graphs?

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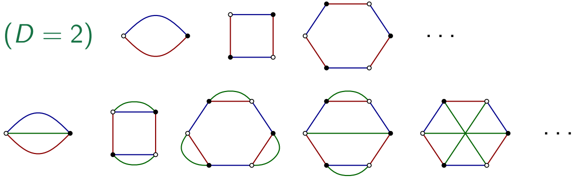
  In contrast to random matrices, answering this question is **hard**.

$(D = 2)$

$(D = 3)$



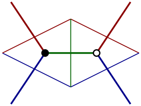
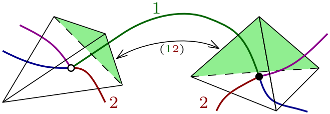
Outline

Large N expansion of matrix models

First generalization: complex colored tensor models

Random geometry applications

Colored triangulations

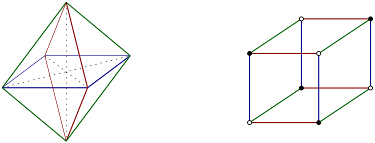
$(D=3)$  

$(D=4)$

Theorem: [Pezzana '74]
D-colored graph $\Leftrightarrow$ triangulation Δ of pseudo-manifold of dim. $D-1$.

Colors $\rightarrow$ unambiguous identification of sub-simplices and their gluings.

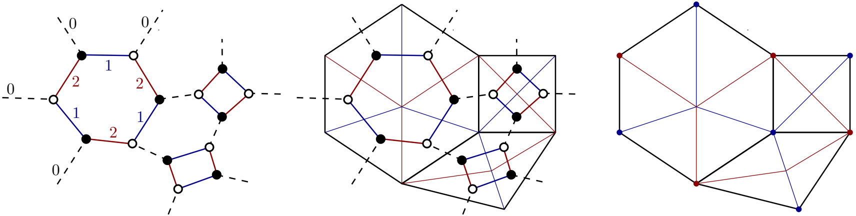
- Bubble $\simeq$ D-colored graph $\simeq$ boundary of D-cell.



- Feynman graph $\simeq$ $(D+1)$-colored graph $\simeq$ Δ of dimension D.

$D = 2$

Gluing of $2p$-angles:

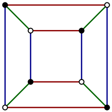


Duality:

3−colored graph	$\longleftrightarrow$	colored triangulation
node	$\longleftrightarrow$	triangle
line	$\longleftrightarrow$	edge
bicolored cycle	$\longleftrightarrow$	vertex

Any orientable surface with boundaries can be represented by such a 3-colored graph.

$D = 3$



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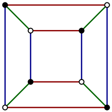

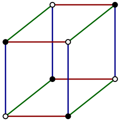

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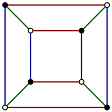

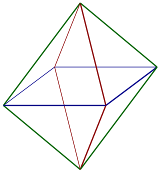

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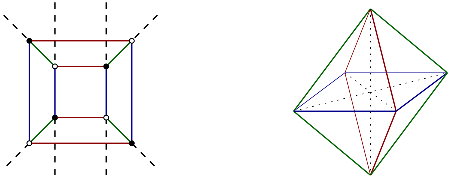

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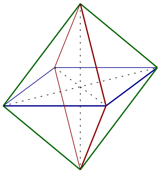

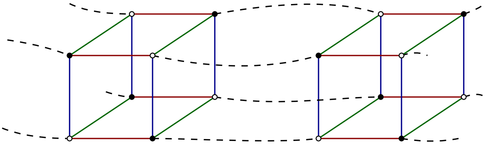

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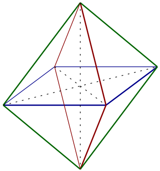
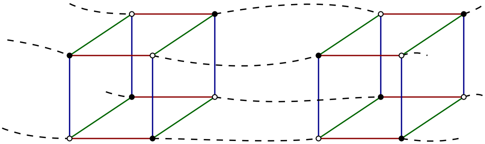
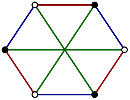




Topological singularities can be generated in $D \geq 3$:



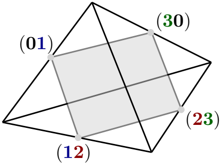
$K_{3,3}$

$g(K_{3,3}) = 1 \Rightarrow$ boundary of a neighborhood not homeomorphic to a ball.

$\rightarrow$ $(D+1)$-colored graphs are dual to *pseudo-manifolds* of dimension D.

$D = 3$

Jackets are dual to embedded quandrangulations in Δ.



$$\sigma = (\mathbf{0123})$$

More precisely, J_σ encodes a *Heegaard splitting* of Δ.

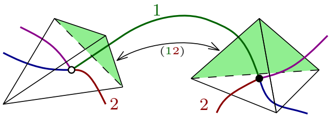
[Ryan '11]

Duality

Colored structure $\Rightarrow$ unambiguous prescription for how to glue D-simplices along their sub-simplices.

$$\begin{array}{rcl} (D+1)-\text{colored graph} & \longleftrightarrow & \text{colored } D-\text{triangulation} \\ \text{node} & \longleftrightarrow & D-\text{simplex} \\ \text{connected component with } k \text{ colors} & \longleftrightarrow & (D-k)-\text{simplex} \end{array}$$

Essential in $D \geq 3$.



$(D = 4)$

Crystallisation theory [Cagliardi, Ferri et al. '80s; Gurau, Ryan '11]

Large-N expansion

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Scaling of bubbles and Feynman expansion governed by Gurau degree ω:

$$\mathcal{F}(\{\lambda_{\mathcal{B}}\}) = \ln \int \mathrm{d}T \, \exp\left(-\overline{T} \cdot T + \sum_{\mathcal{B}} \lambda_{\mathcal{B}} \, N^{-\frac{2}{(D-2)!}\omega(\mathcal{B})} \, \mathrm{Tr}_{\mathcal{B}}(\overline{T}, T) \right)$$

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where $$\boxed{\omega(\Delta) = D - n_{D-2}(\Delta) + \frac{D(D-1)}{4} n_D(\Delta)}$$

- $\omega \in \mathbb{N}$
- generalization of the genus: $D = 2 \Rightarrow \omega = g$
- *not* a topological invariant of Δ when $D \geq 3$
- however: $\omega = 0 \Rightarrow \Delta$ is a D-sphere

Botanical interlude: melon diagrams

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Melonic theories $\rightarrow$ Feynman expansion dominated by ***melon diagrams***:

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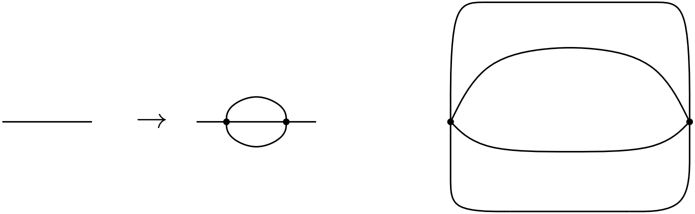

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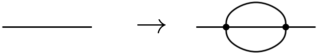

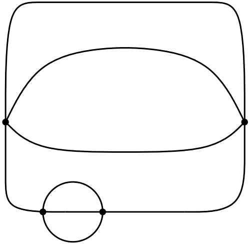

Botanical interlude: melon diagrams

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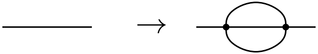

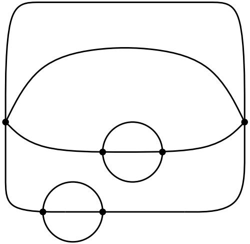

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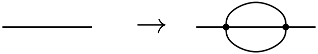

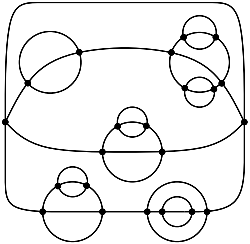

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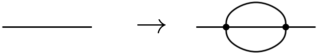
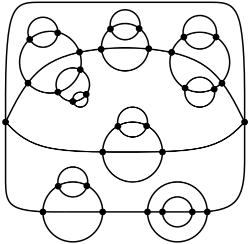
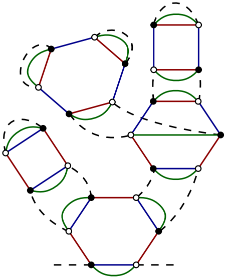




LEADING ORDER

[BONZOM, GURAU, RIELLO, RIVASSEAU '11;...]



$\omega(\Delta) = 0 \quad \Leftrightarrow \quad \Delta$ is melonic

$\rightarrow$ special triangulations of the D-sphere, with a tree-like combinatorial structure.

Closed equation for their generating function:

$$\boxed{G(\lambda) = 1 + \lambda G(\lambda)^{D+1}} \qquad \text{(Fuss-Catalan)}$$

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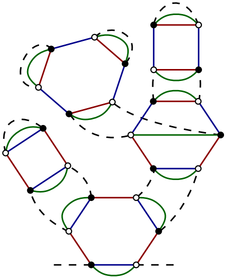
[Bonzom, Gurau, Riello, Rivasseau '11;...]



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Critical behaviour:

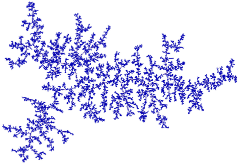
$$G(\lambda_c) - G(\lambda) \underset{\lambda \to \lambda_c}{\sim} K\,(\lambda_c - \lambda)^{1/2}$$

$$\Leftrightarrow \quad \#\{\text{ rooted melonic } \Delta\,\} \sim K \lambda_c^{-n_\Delta} {n_\Delta}^{-3/2}$$

Universal critical exponent $3/2$ associated to combinatorial trees.

CONTINUUM LIMIT [GURAU, RYAN '13]

Melons are branched polymers
i.e. they converge to the continuous random tree [Aldous '91].



Credit: I. Kortchemski (https://igor-kortchemski.perso.math.cnrs.fr/images.html)

$$\#\{\text{ rooted melonic } \Delta\,\} \sim K \lambda_c^{-n_\Delta} {n_\Delta}^{-3/2}$$

$\boxed{d_{\text{spectral}} = 4/3}$; distance scale $\sim {n_\Delta}^{1/2}$ and $d_{\text{Hausdorff}} = 2$

$\Rightarrow$ strong universality: limit independent of D!

Further results

- Combinatorial classification of graphs at order $\omega > 0$:
  "it's melons all the way down". [Gurau, Schaeffer '13]
- Double-scaling. [Bonzom, Gurau, Kaminski, Dartois, Oriti, Ryan, Tanasa '13 '14]
- Schwinger-Dyson eq. $\rightarrow$ analogue of loop equations. [Gurau '11]
- Non-perturbative treatment. [Gurau '14]
- ...
- Applications in Group Field Theory:
  [Boulatov, Ooguri, '92... Freidel, Gurau, Oriti '00s '10s...]

  Melonic behaviour $\Rightarrow$ rigorous renormalization theorems
  [Ben Geloun, Rivasseau '11; SC, Oriti, Rivasseau '13;...]
  [Review SC '16]

Beyond branched polymers?

No-go:

- Non-melonic large-N limits have been explored.
  [Bonzom, Delpouve, Rivasseau '15; Bonzom, Lionni '16; Lionni, Thüringen '17]
- Universality theorem: $D = 3 \Rightarrow$ branched polymers for arbitrary spherical bubbles. [Bonzom '18]

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Yes go?

- D even $\Rightarrow$ Brownian sphere, branched polymers and mixtures.
  [Bonzom, Delpouve, Rivasseau '15]
- Simple combinatorial restrictions may change the universality class:

$$\text{branched polymers} \underset{\text{2PI}}{\longrightarrow} \text{Ising on a random surface}$$

[Benedetti, SC, Toriumi, Valette '20]

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Major open question:
genuinely new random geometric phase suitable for QG in $D \geq 3$?
[Lionni, Marckert '19]

Summary

Tensor models for random geometry:

- well-defined generalization of the matrix models approach;
- reproduce previously known universality classes: continuous random tree, Brownian sphere, and mixtures;
- tend to be dominated by tree-like combinatorial species $\Rightarrow$ no genuinely new universality class discovered so far...

  ...but a vast parameter space remains to be explored.

Entry points into the literature:

- "Random tensors", Gurau, 2016;
- "The Tensor Track" I-IV, Rivasseau, 2011-2016;
- "Colored Discrete Spaces", Lionni, 2018.