TENSOR NETWORKS FOR QUANTUM MANY-BODY SYSTEMS

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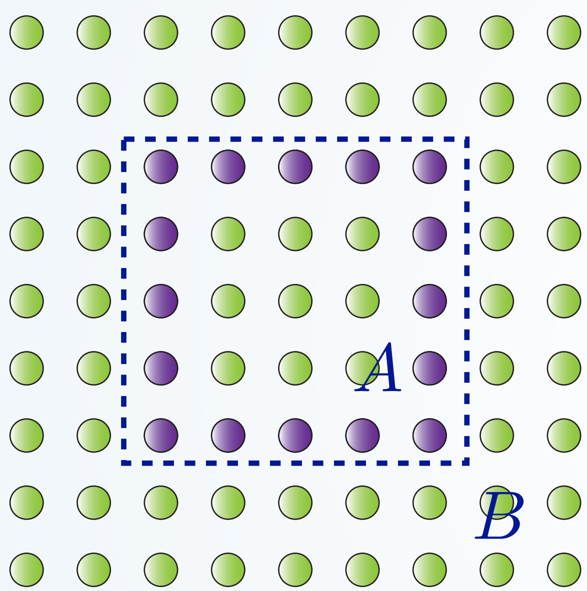
(Garching)

CIRM March 2022

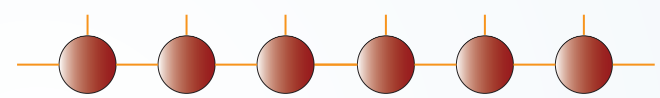
In first session...

TNS = entanglement based ansatz

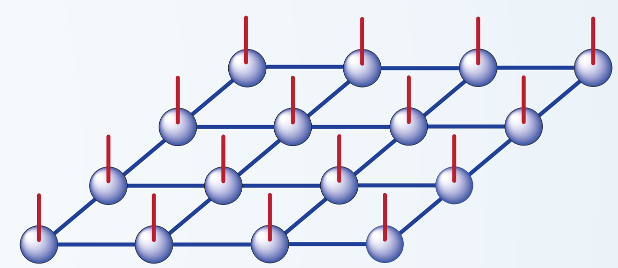
area law



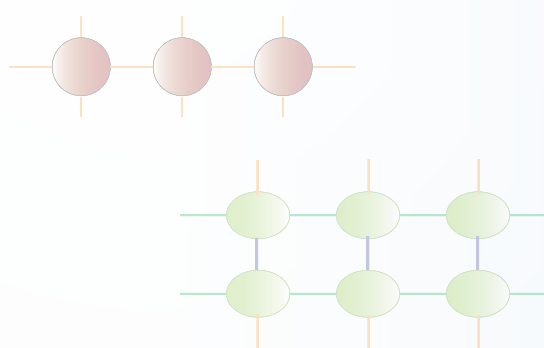
MPS



PEPS



MPO



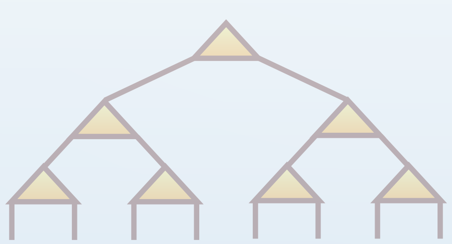
Schollwöck Ann.Phys.2011

Verstraete et al. Adv. Phys. 2008

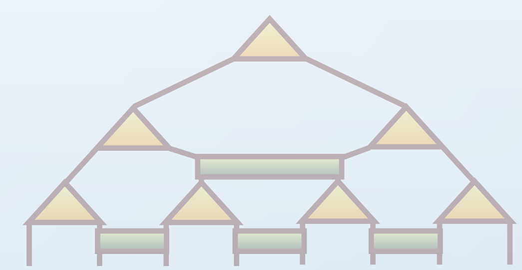
Orús, Ann. Phys. 2014

other TNS

TTN



MERA



MPS PROPERTIES

Matrix Product States

RECAP

good approximation of ground states

Verstraete, Cirac, PRB 2006

Hastings, J. Stat. Phys 2007

gapped finite range Hamiltonian $\Rightarrow$
area law (ground state)

Cramer, Eisert, Plenio, RMP 2009

efficient calculation of expectation values

exponentially decaying correlations

can be prepared efficiently

PEPS PROPERTIES

Projected Entangled Pairs States

RECAP

no efficient calculation of expectation values

can hold algebraically decaying correlations

cannot be prepared efficiently

ground state of local frustration-free Hamiltonians

efficient approximation of thermal states

Hastings PRB 2006
Molnar et al PRB 2015

In this session…

other TNS

basic algorithms

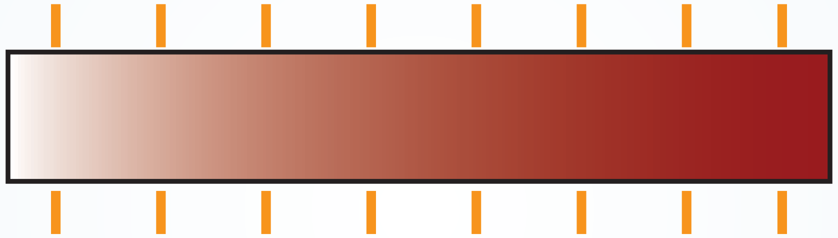
MPO

Matrix Product Operators

MPO

Matrix Product Operator

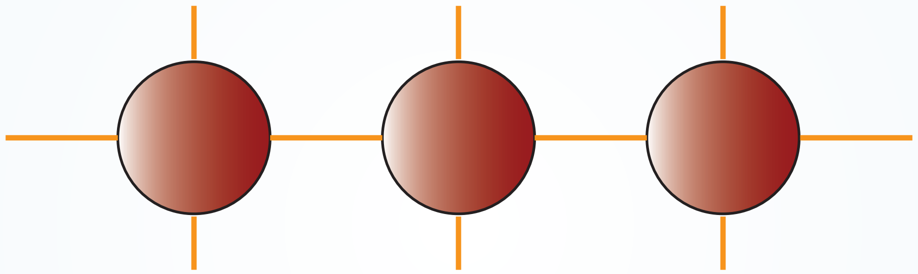
Same kind of ansatz for operators



MPO

Matrix Product Operator

Same kind of ansatz for operators



$$\hat{M} = \sum_{i_1, j_1 \dots i_N, j_N} \mathrm{tr}(M_1^{i_1 j_1} M_2^{i_2 j_2} \dots M_N^{i_N j_N}) |i_1 \dots i_N\rangle\langle j_1 \dots j_N|$$

MPO is an operator with MPS form in the chosen basis!

Verstraete et al., PRL 2004
Zwolak, Vidal, PRL 2004
Pirvu et al., NJP 2010

MPO

Matrix Product Operator

local Hamiltonians are simple MPOs

finite state automata ⟶ recognize regular expressions

$(0*)1(0*)$

accept	reject
1000	0000
0100	1100
0010	1010
0001	...

$$|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle$$

$$\sigma_x \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_x \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \sigma_x \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1} \otimes \sigma_x = \sum_i \sigma_x^{[i]}$$

MPO

local Hamiltonians are simple MPOs

finite state automata ⟶ recognize regular expressions

FSA = computational model

S states $S_0, S_f \in S$

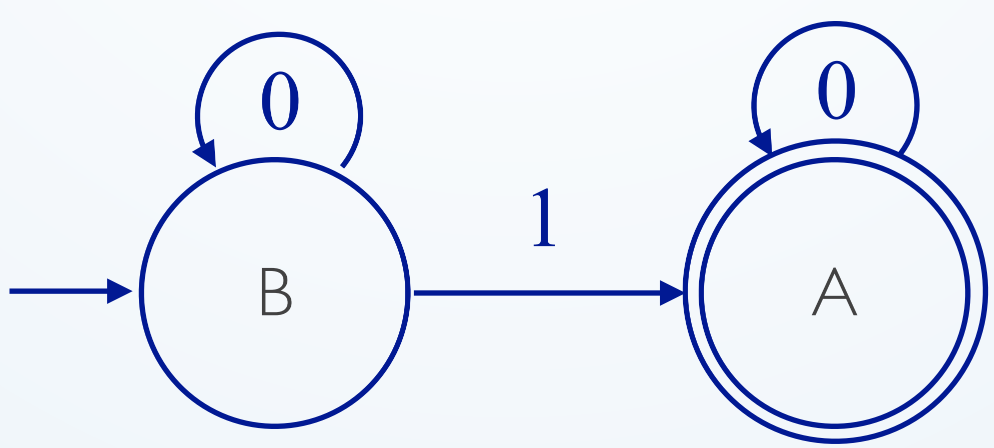
B: before 1

A: after 1

Σ input alphabet 0, 1

$S \times \Sigma \to S$ transitions

$$(0*)1(0*)$$

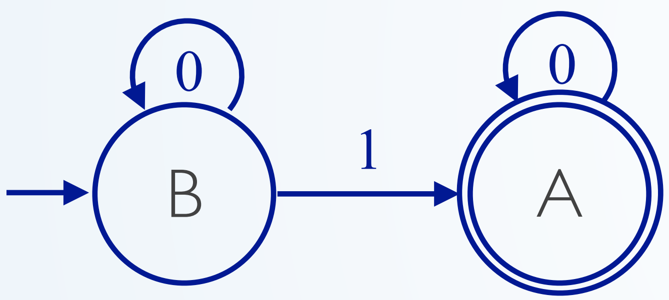


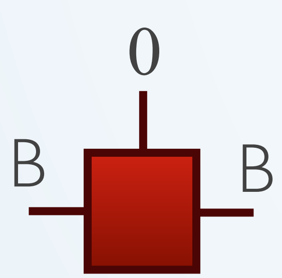
MPO

Matrix Product Operator

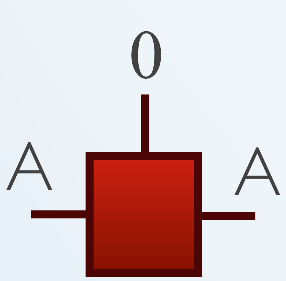
local Hamiltonians are simple MPOs

translate to MPS/MPO → input symbols = physical indices
nr of states = bond dimension
valid transitions = non-vanishing tensor elements





$= 1$



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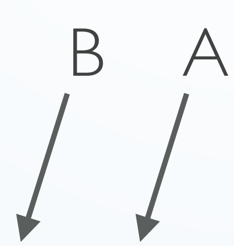
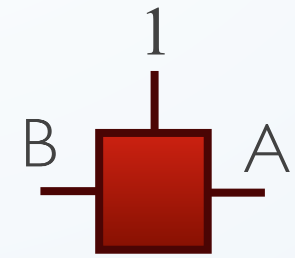


$$M^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



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$$M^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

boundaries M_L, M_R

MPO

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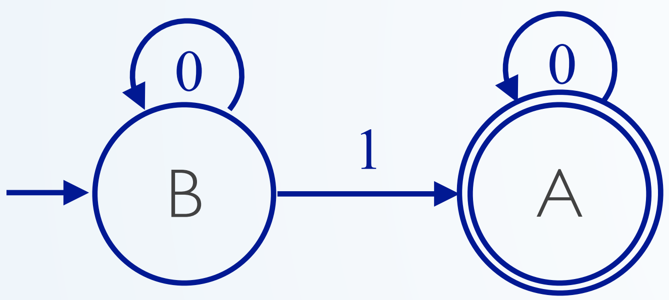

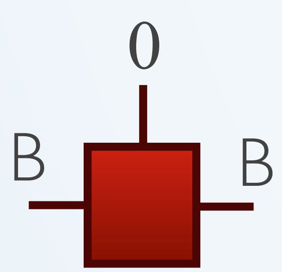

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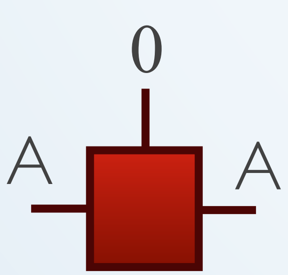

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B A

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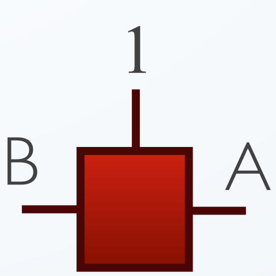

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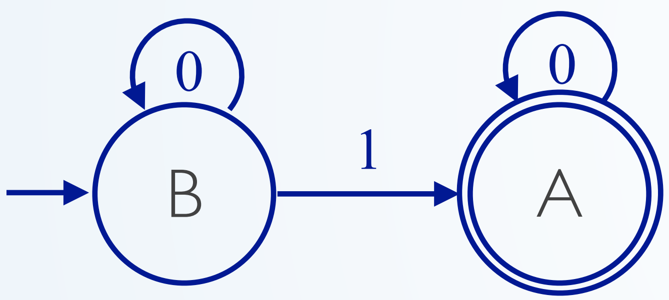

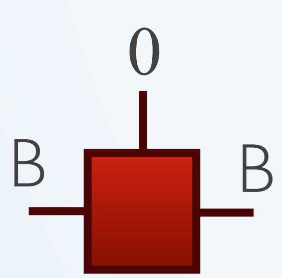

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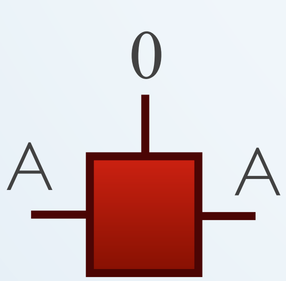

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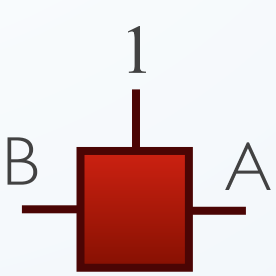


$= 1$

$$M^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

boundaries M_L, M_R

MPO

Matrix Product Operator

local Hamiltonians are simple MPOs

$$M^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad M^1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\sum_{\{i_k\}} M_L^{i_1} M^{i_2} \dots M^{i_{N-1}} M_R^{i_N} |i_1 \dots i_N\rangle \qquad i_k = 0, 1$$

expressed as operator(vector) valued matrix

$$M = \begin{pmatrix} |0\rangle & |1\rangle \\ 0 & |0\rangle \end{pmatrix} \qquad M = \begin{pmatrix} \mathbb{1} & \sigma_x \\ 0 & \mathbb{1} \end{pmatrix}$$

$$|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle \qquad \sum_i \sigma_x^{[i]}$$

MPO for Ising Hamiltonian

$$H = J \sum \sigma_z^{[i]} \sigma_z^{[i+1]} + g \sum \sigma_x^{[i]}$$

$$\sum_{\{i_k\}} M_L^{i_1} M^{i_2} \dots M^{i_{N-1}} M_R^{i_N} |i_1 \dots i_N\rangle \qquad i_k = \mathbb{1},\, \sigma_x,\, \sigma_z$$

M_L

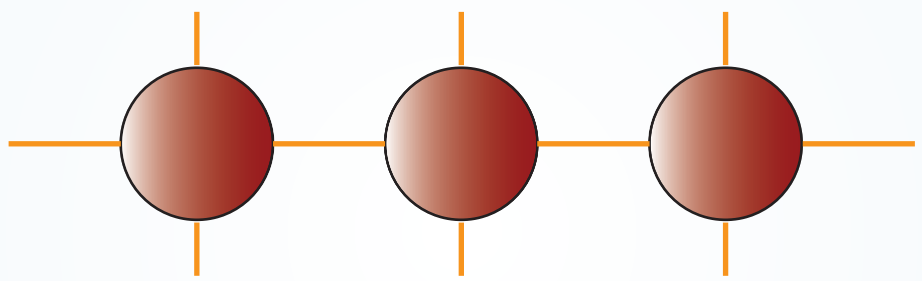
$$M = \begin{pmatrix} \mathbb{1} & J\sigma_z & g\sigma_x \\ 0 & 0 & \sigma_z \\ 0 & 0 & \mathbb{1} \end{pmatrix}$$

M_R

MPO

Matrix Product Operator

an ansatz for density matrices



need some properties

$$\rho = \sum_{i_1, j_1 \dots i_N, j_N} \mathrm{tr}(M_1^{i_1 j_1} M_2^{i_2 j_2} \dots M_N^{i_N j_N}) |i_1 \dots i_N\rangle\langle j_1 \dots j_N|$$

can we impose them *locally*?

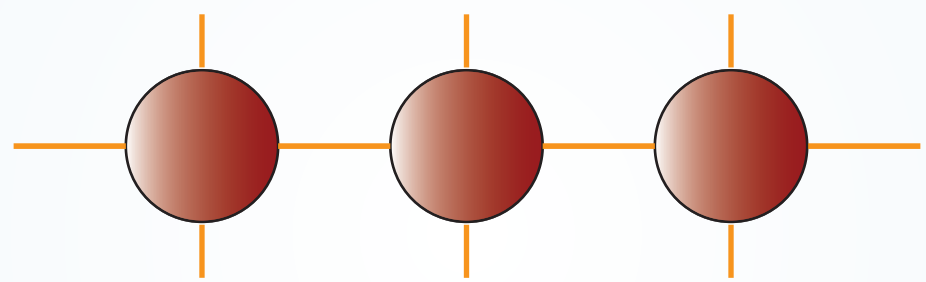
$\rho = \rho^\dagger$ $\qquad$ $\mathrm{tr}\rho = 1$ $\qquad$ $\rho \geq 0$

not all MPO satisfy them!

MPO

Matrix Product Operator

an ansatz for density matrices



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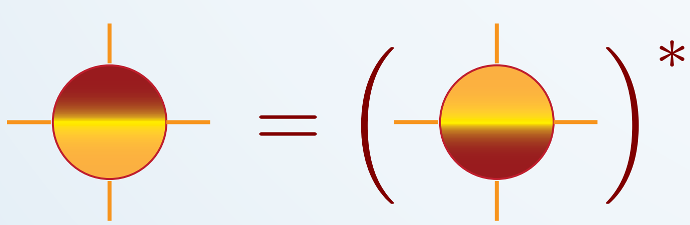

MPO

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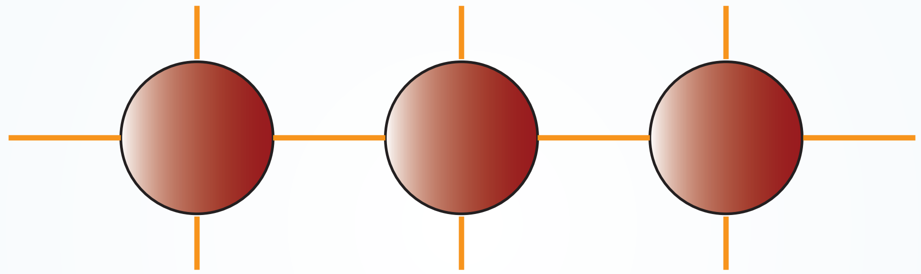

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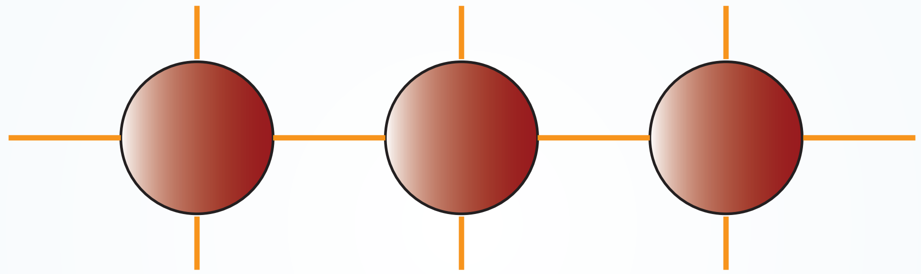

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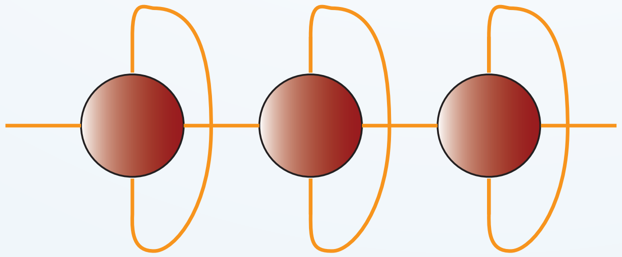
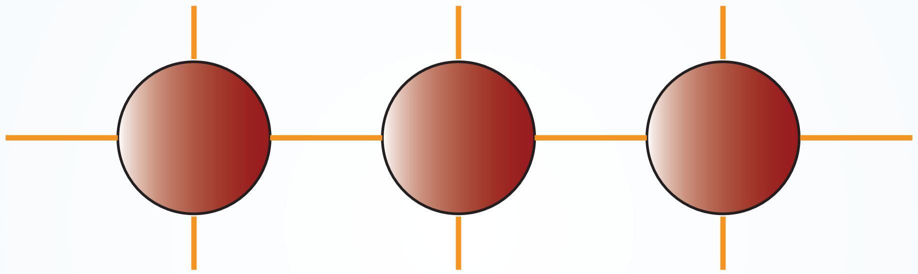
 $= 1$

MPO

Matrix Product Operator

an ansatz for density matrices



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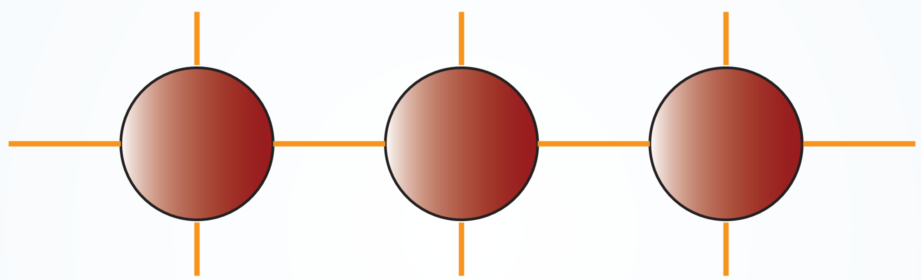
an ansatz for density matrices



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can we impose them *locally*?

$\rho = \rho^\dagger$ ✓

$\mathrm{tr}\rho = 1$ ✓

$\rho \geq 0$ ✗

there is a way

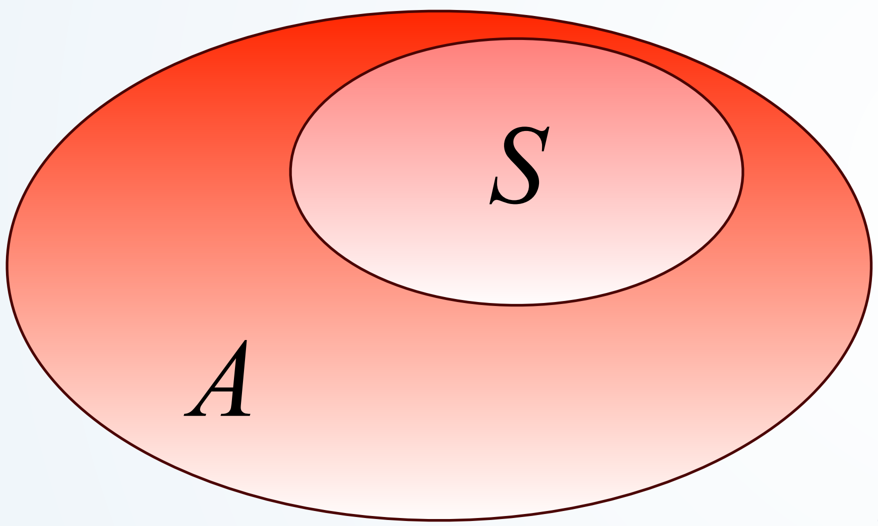
Kliesch et al., PRL 2014

de las Cuevas et al. J Math Phys 2016

MPO

Matrix Product Operator

purification



$$\rho_S = \sum_i \lambda_i |\varphi_i\rangle\langle\varphi_i|$$

$$0 \leq \lambda_i \leq 1$$

$$\sum_i \lambda_i = 1$$

ancillary system **A** $\qquad d_A \leq d_S$

$$|\Psi\rangle_{SA} = \sum_i \sqrt{\lambda_i} |\varphi_i\rangle_S \otimes |i\rangle_A$$

$$\rho_S = \mathrm{tr}_A \left(|\Psi_{SA}\rangle\langle\Psi_{SA}| \right)$$

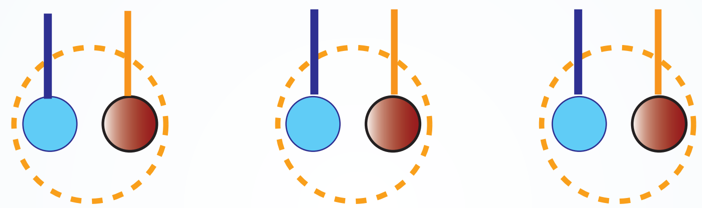
orthogonal

unitary freedom on ancilla

MPO

Matrix Product Operator

purification



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Verstraete et al., PRL 2004

Zwolak and Vidal, PRL 2004

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Matrix Product Operator

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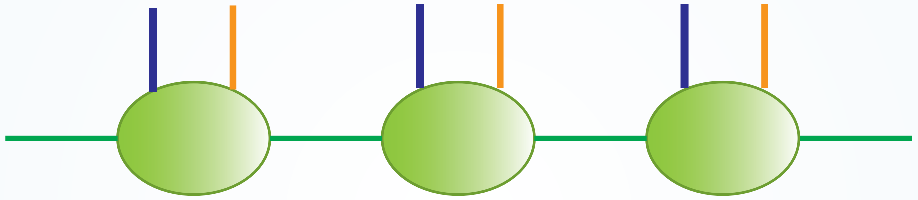

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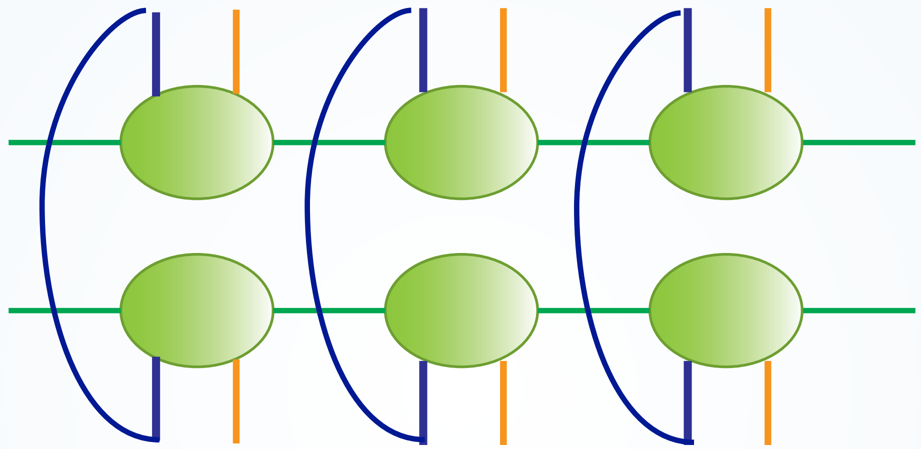

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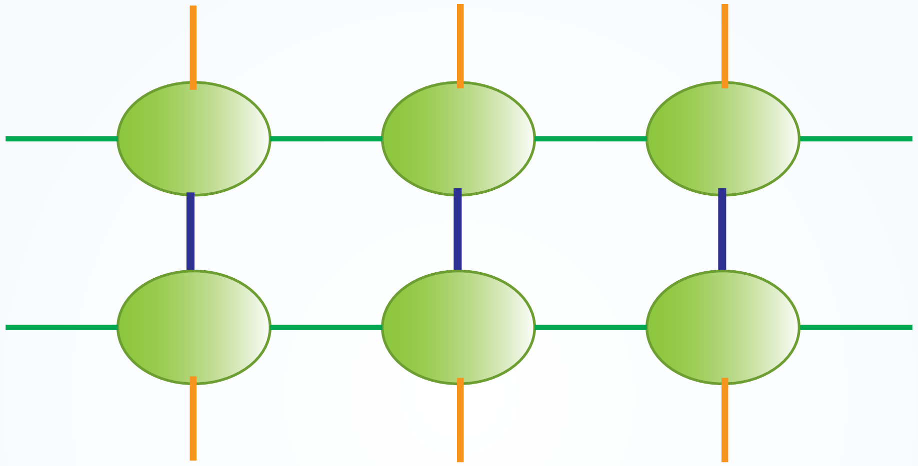

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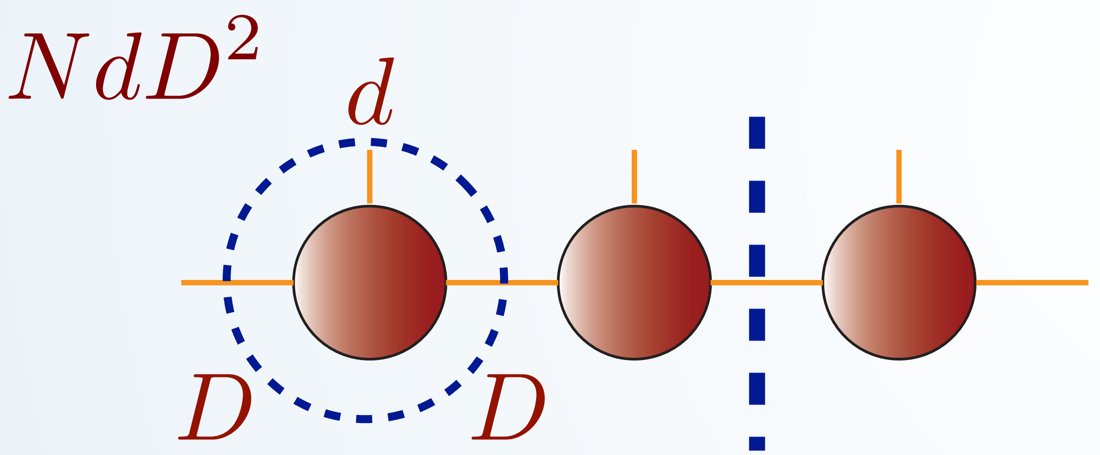
Verstraete et al., PRL 2004

Zwolak and Vidal, PRL 2004

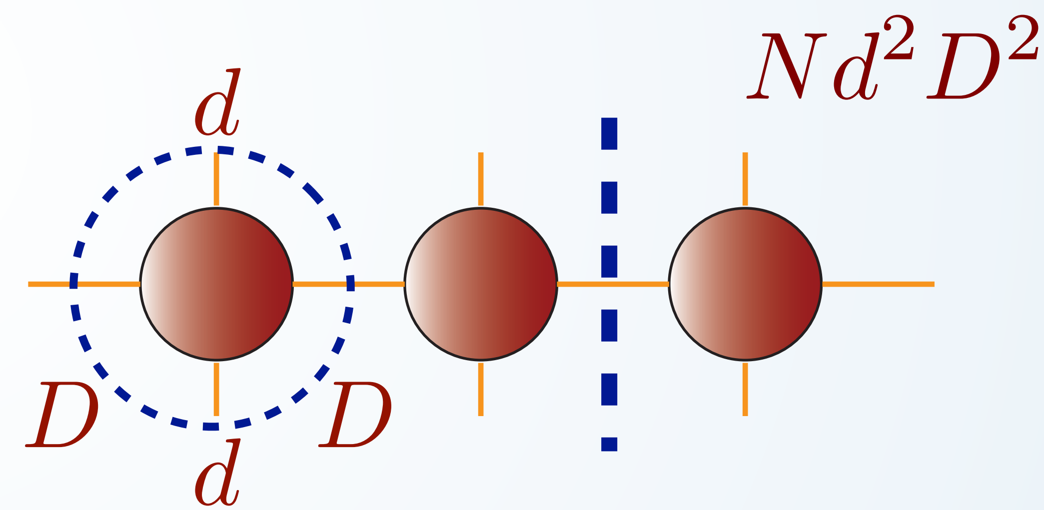
MPO

Matrix Product Operator

Bond dimension determines number of parameters



vs



Schmidt rank → entanglement

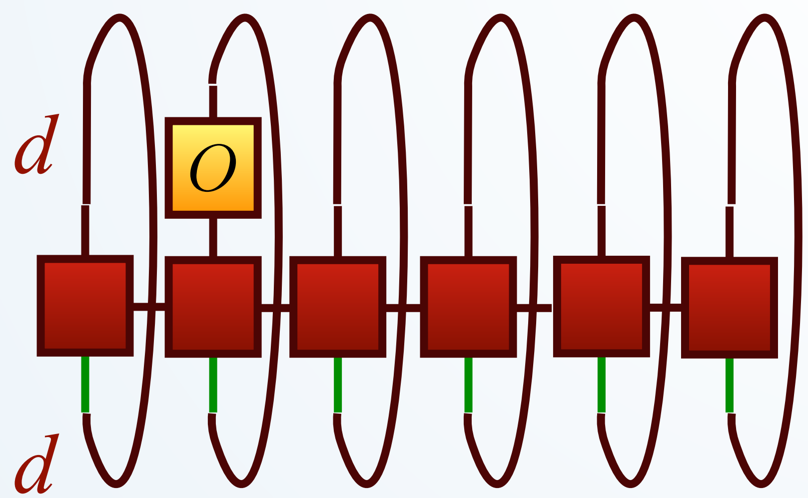
Schmidt rank in operator space → operator space entanglement entropy

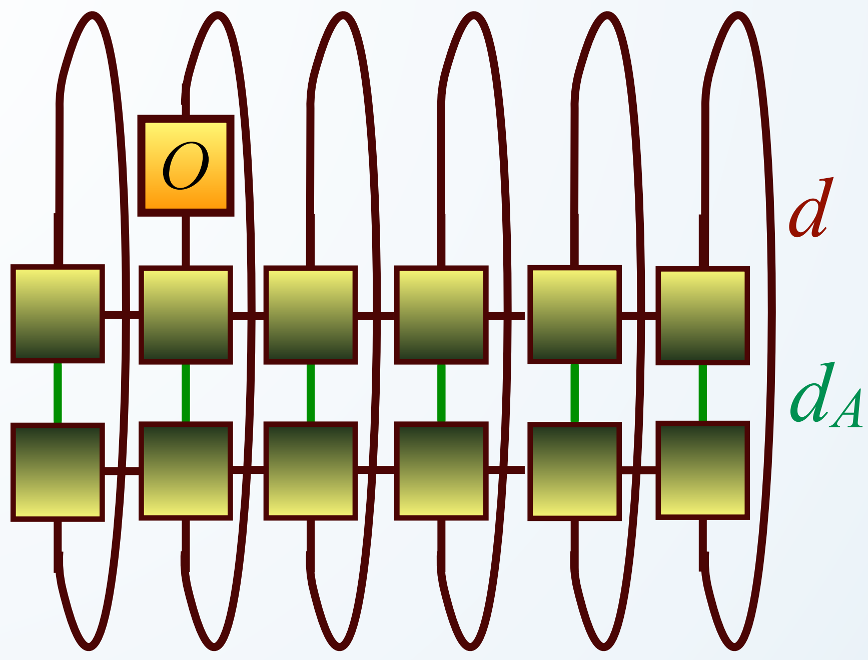
MPO

Matrix Product Operator

expectation values

$$\langle O \rangle_\rho = \mathrm{tr}(O\,\rho) = \mathrm{tr}_S[O\,\mathrm{tr}_A(|\Psi\rangle\langle\Psi|)] = \mathrm{tr}(O\,|\Psi\rangle\langle\Psi|)$$





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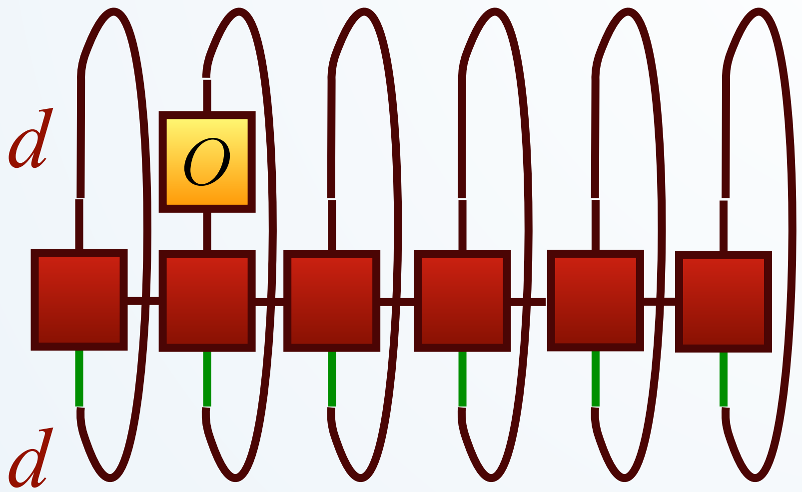
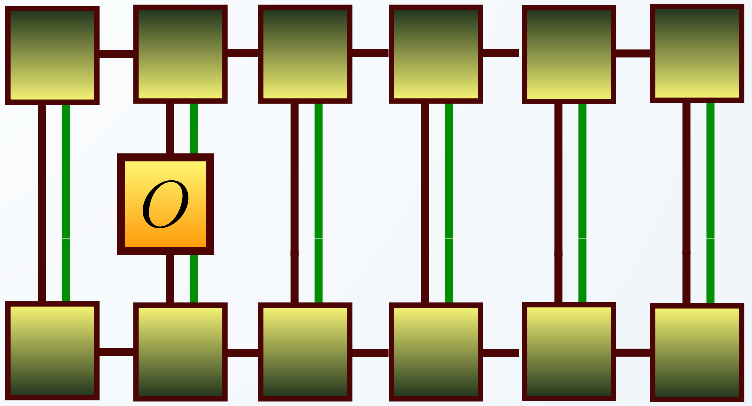
$$= \langle\Psi|O|\Psi\rangle$$



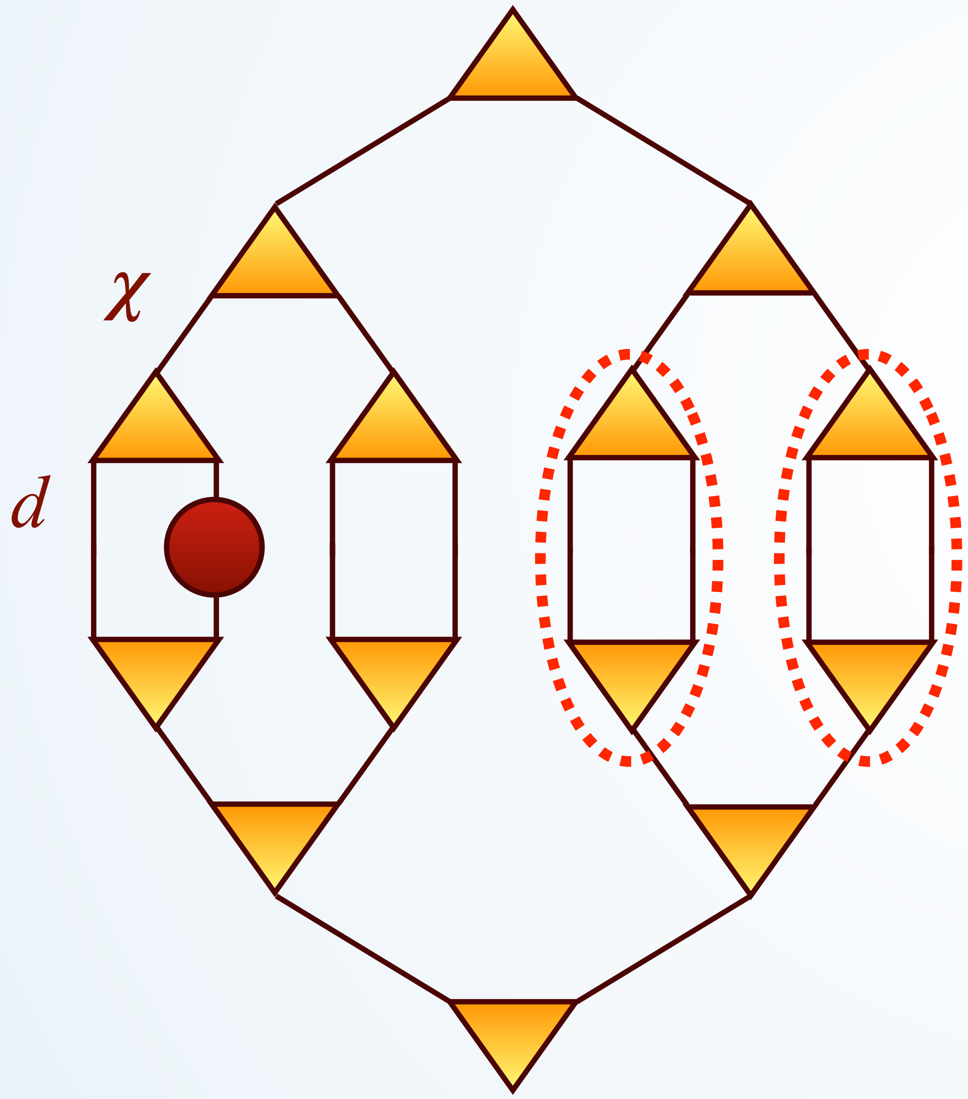


OTHER TNS

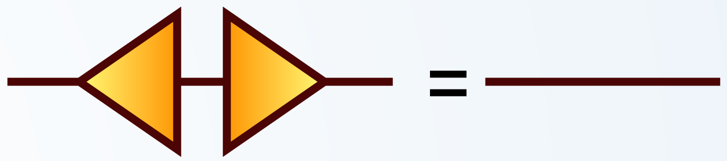
not fulfilling area law

TTN

Tree Tensor States



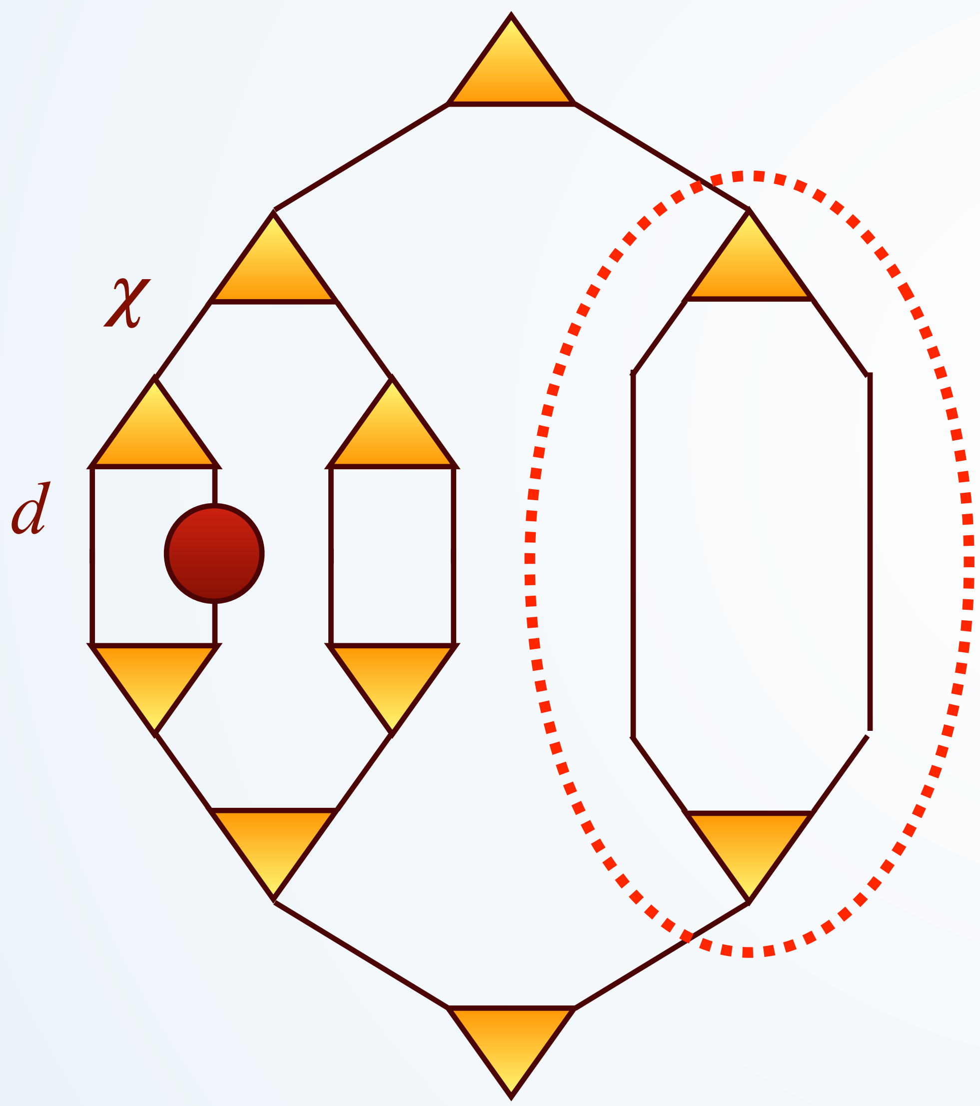
efficient contraction

isometries 

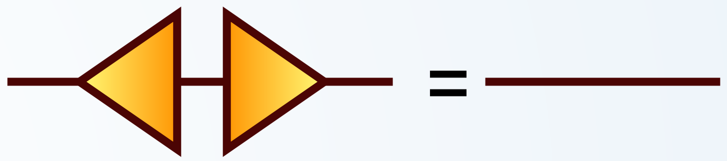
Shi et al PRA (2006)
Tagliacozzo et al PRB80 (2009)
Silvi et al PRA81 (2010)

TTN

Tree Tensor States



efficient contraction

isometries 

Shi et al PRA (2006)
Tagliacozzo et al PRB80 (2009)
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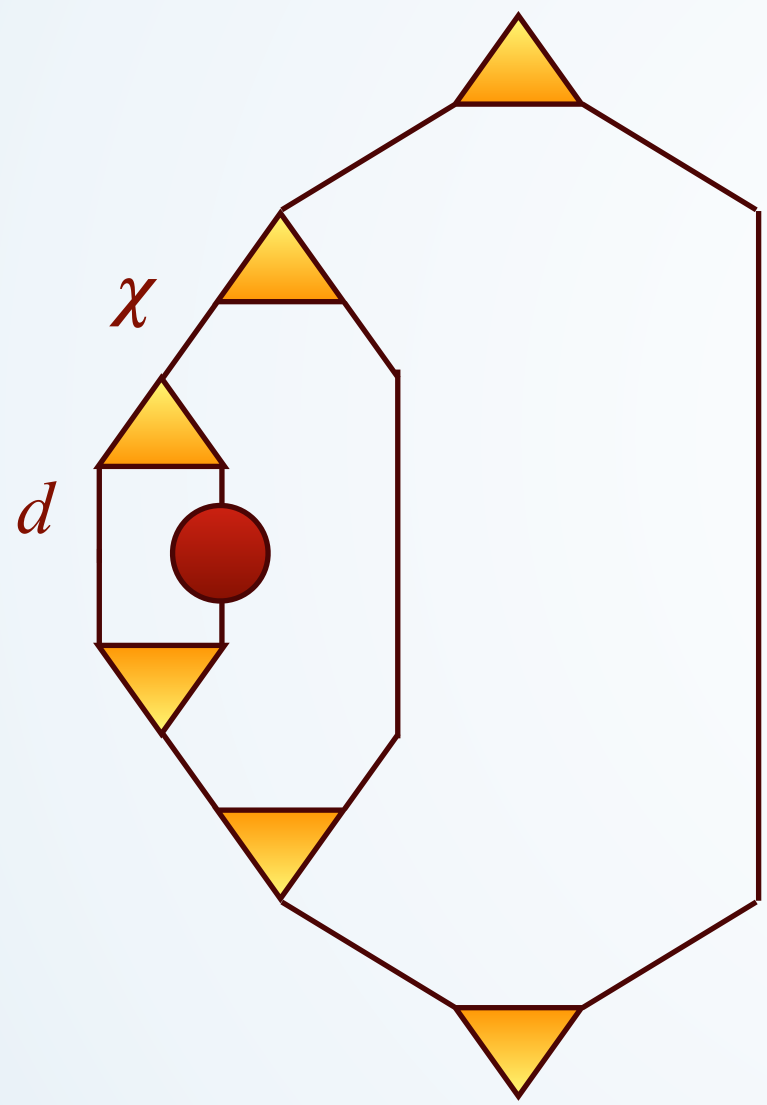
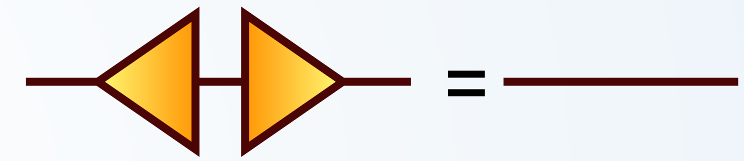
TTN

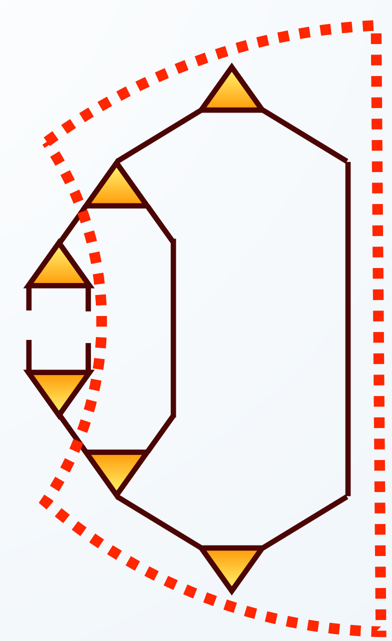
Tree Tensor States



efficient contraction

isometries 

reduced density matrix

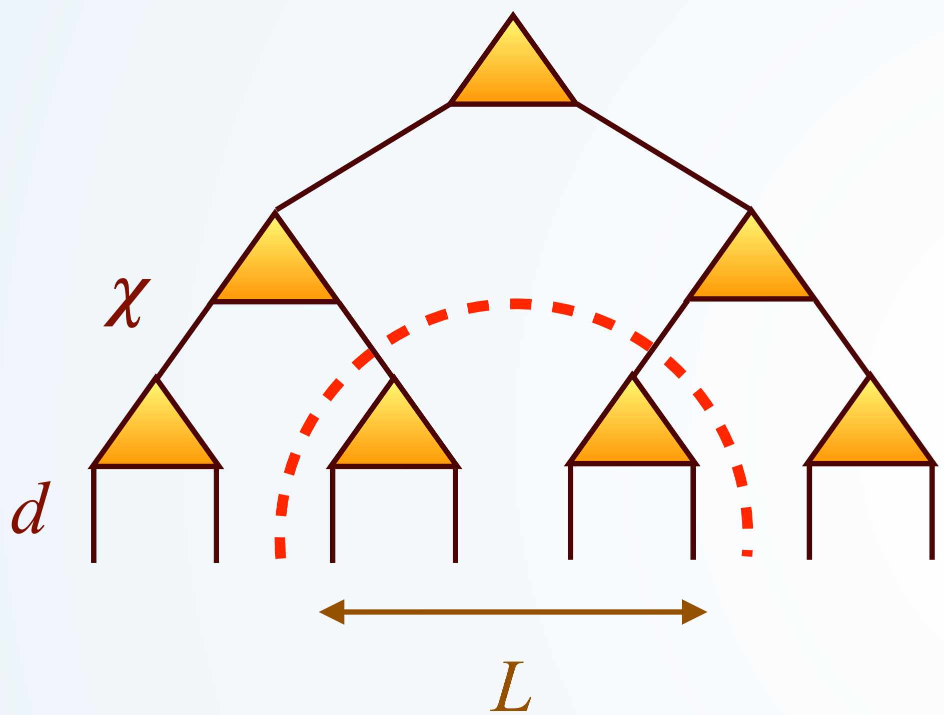


the rank of
this matrix
determines
the max
entanglement

Shi et al PRA (2006)
Tagliacozzo et al PRB80 (2009)
Silvi et al PRA81 (2010)

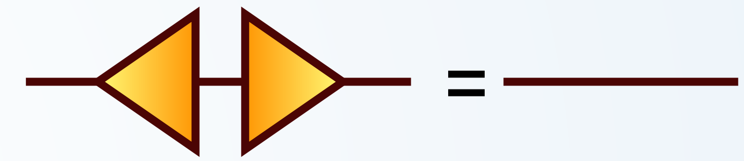
TTN

Tree Tensor States



efficient contraction

isometries



can be $\log L$

homogeneous case: average correlations decay as power law

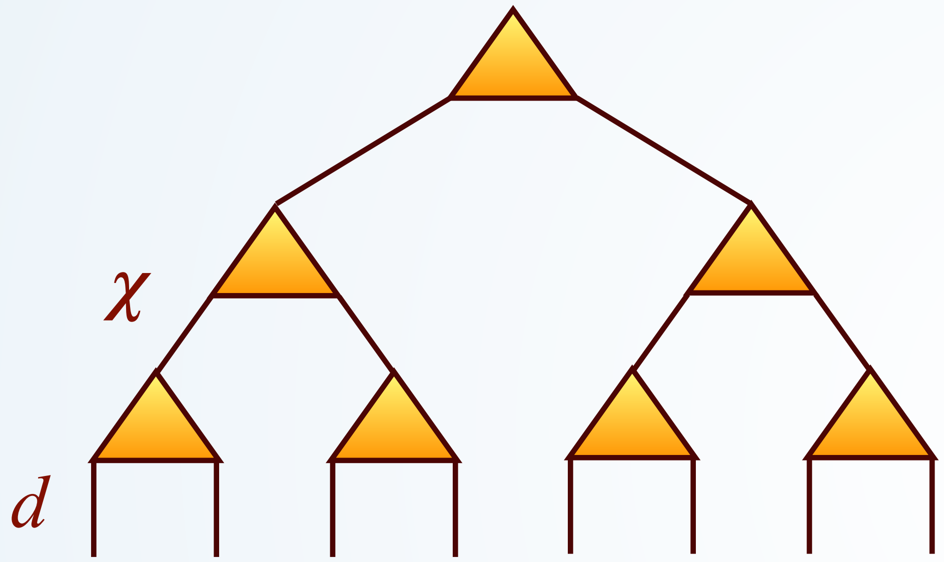
works also in two dimensions, or PBC

because no loops ⇒ canonical form

Shi et al PRA74 (2006)
Tagliacozzo et al PRB80 (2009)
Silvi et al PRA81 (2010)

MERA

Multiscale Entanglement Renormalization Ansatz

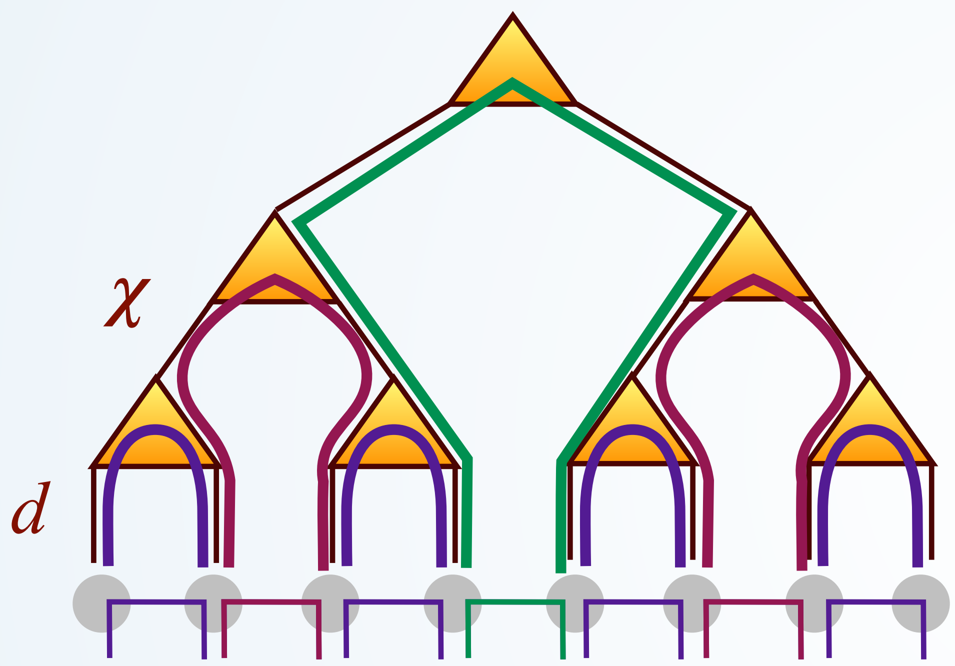


tree as real space renormalization

Vidal PRL99 (2007)

MERA

Multiscale Entanglement Renormalization Ansatz



tree as real space renormalization

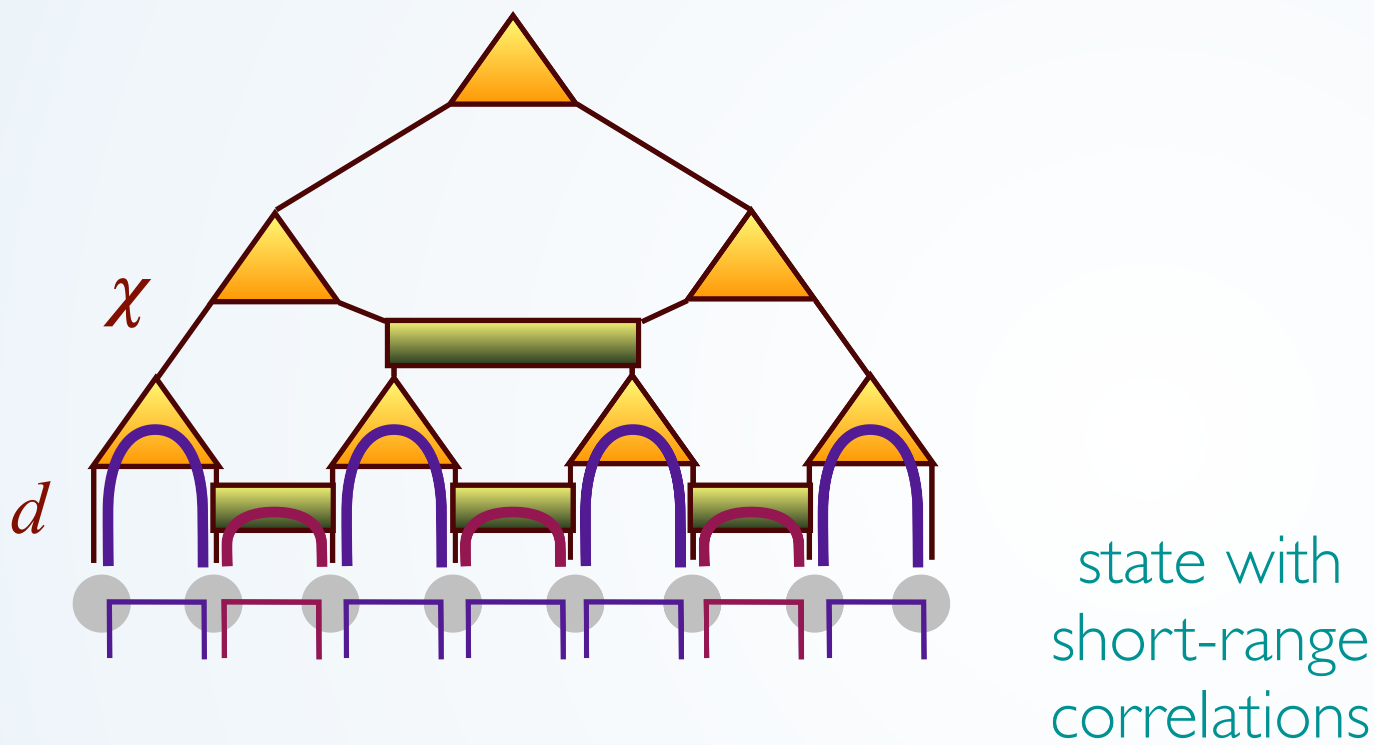
state with
short-range
correlations

cannot get rid of short
range correlations

Vidal PRL99 (2007)

MERA

Multiscale Entanglement Renormalization Ansatz

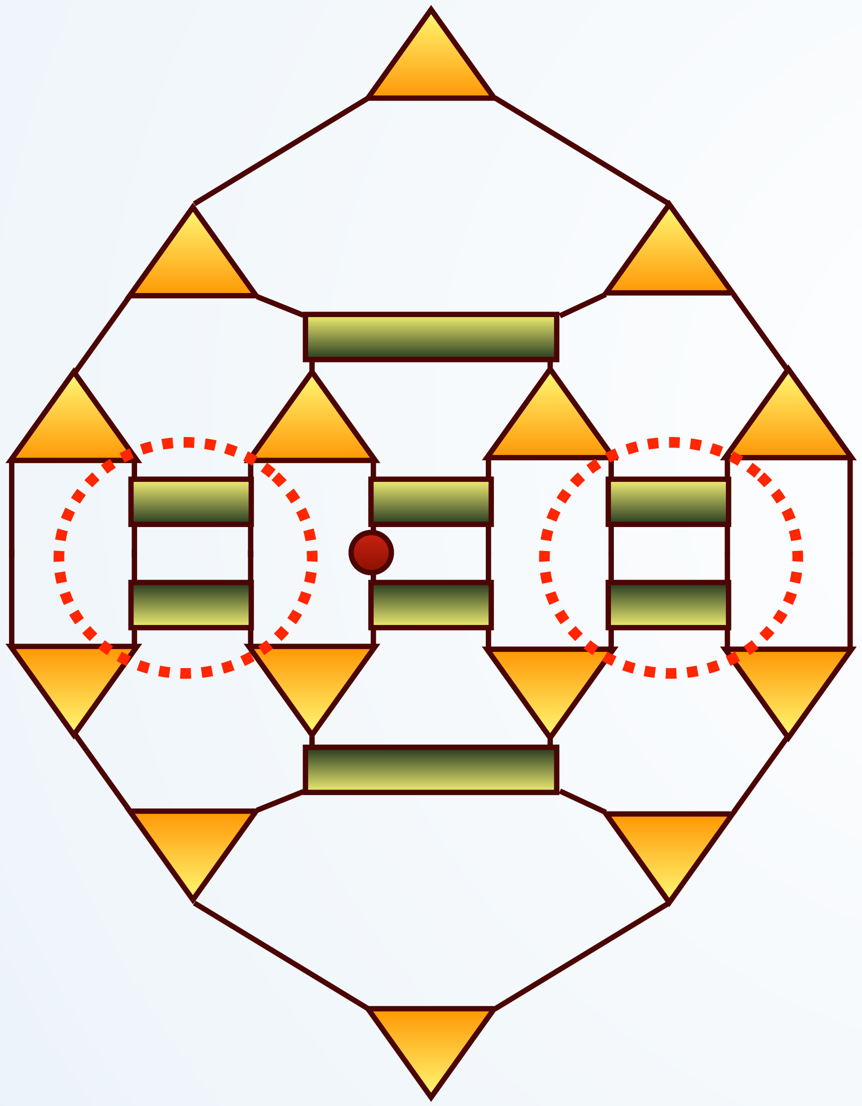


disentanglers

MERA = Q circuit to prepare the state

MERA

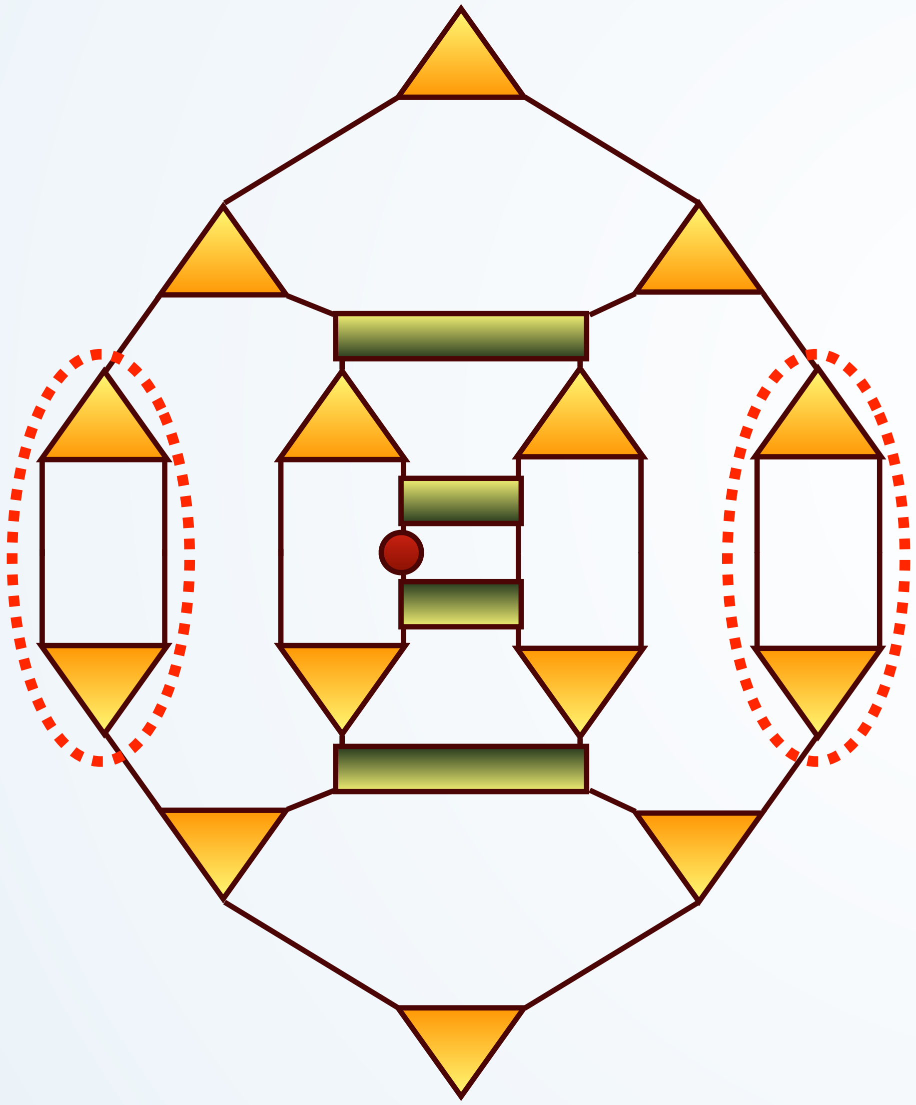
Multiscale Entanglement Renormalization Ansatz



efficient contraction

MERA

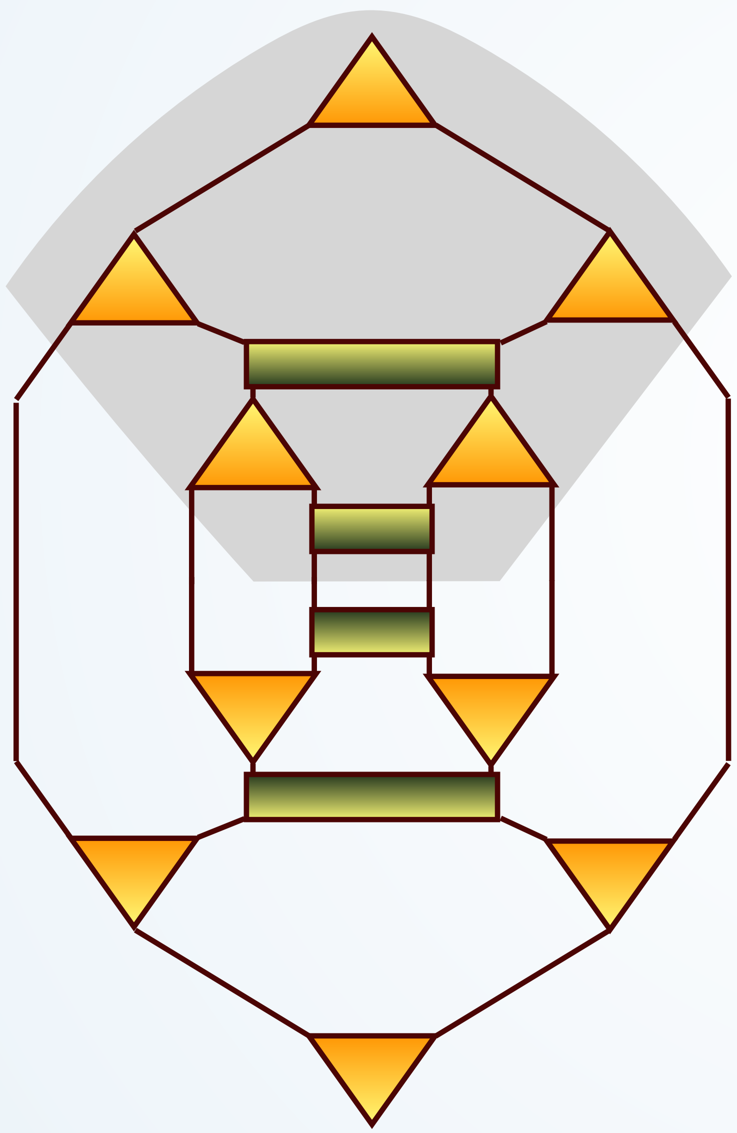
Multiscale Entanglement Renormalization Ansatz



efficient contraction

MERA

Multiscale Entanglement Renormalization Ansatz



efficient contraction

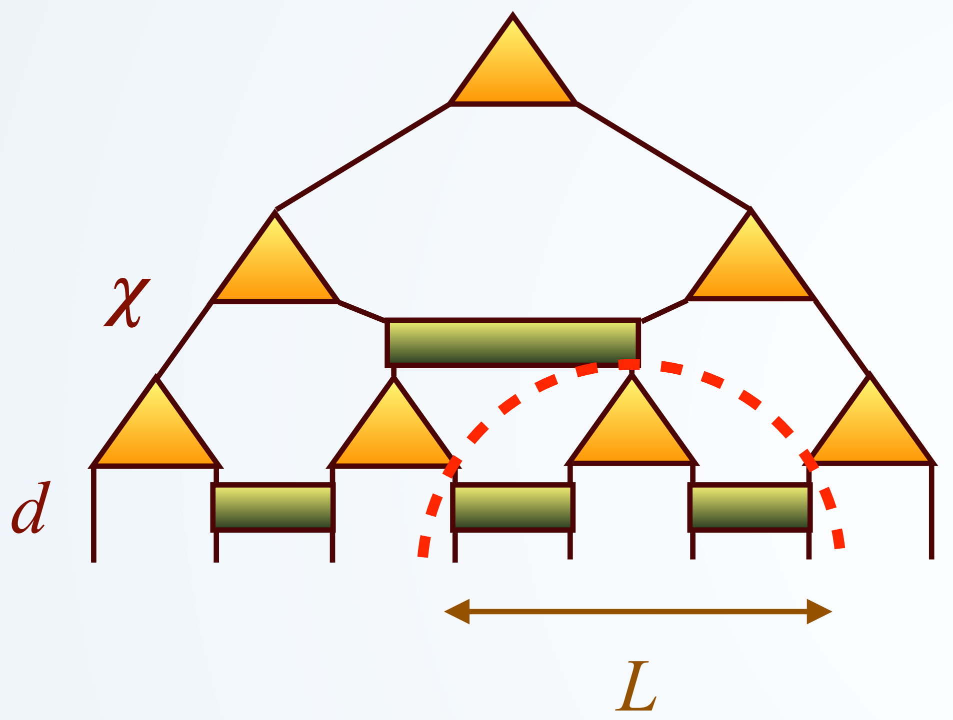
causal cone

bounded width

$\log L$ height

MERA

Multiscale Entanglement Renormalization Ansatz



efficient contraction

causal cone

bounded width
$\log L$ height

$$S(L) \leq \log \chi \log L$$

logarithmic violation of area law
can describe critical systems

in 2D MERA are a subset of PEPS

WHAT CAN WE DO WITH TNS?

BASIC ALGORITHMS

two main types

variational optimization

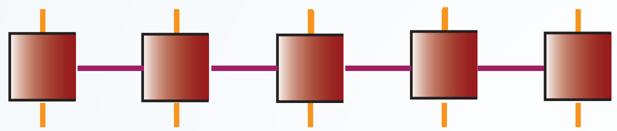
used for ground states

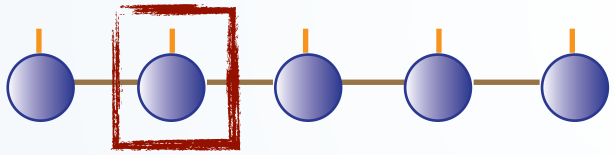
applying a (local) operator

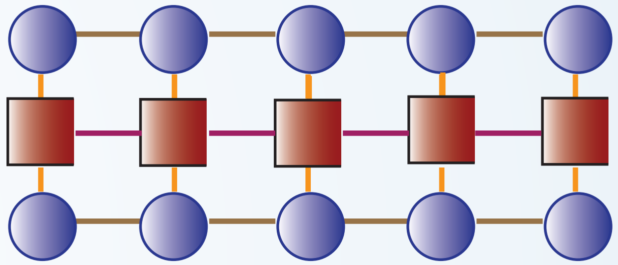
basis of most common
evolution algorithms

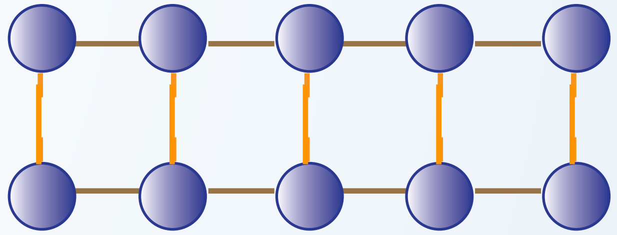
BASIC ALGORITHMS

variational minimization of energy

$H =$ 

$|E_0\rangle \simeq$ 

$\langle \Psi | H | \Psi \rangle$ 

$\langle \Psi | \Psi \rangle$ 

Variational principle

$$\min_{\{A\}} \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} \longrightarrow \min_{A} \frac{\bar{A} H_{\text{eff}} A}{\bar{A} N_{\text{eff}} A}$$

White, PRL 1992

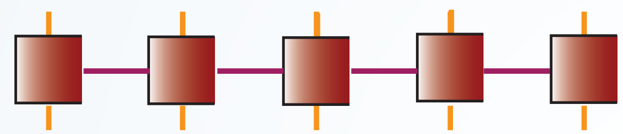
Verstraete, Porras, Cirac, PRL 2004

Schollwöck, RMP 2005, Ann. Phys. 2011

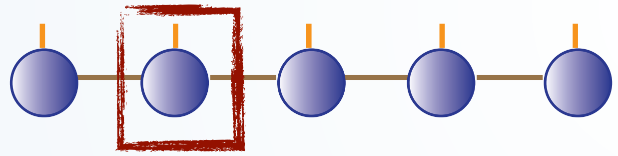
BASIC ALGORITHMS

variational minimization of energy

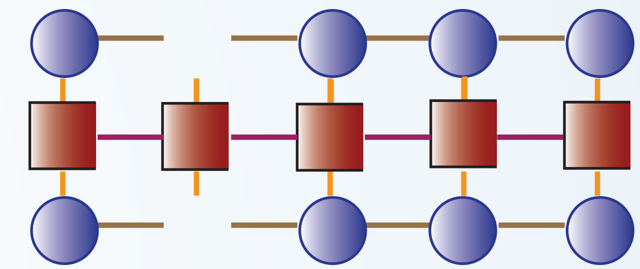
$H =$



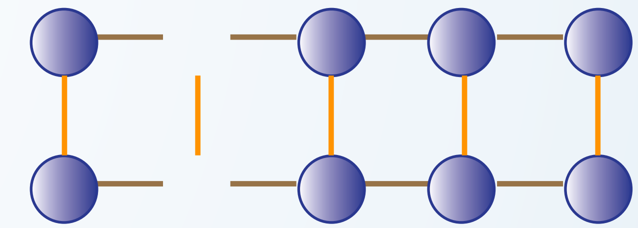
$|E_0\rangle \simeq$



H_{eff}



N_{eff}



Variational principle

$$\min_{\{A\}} \frac{\langle \Psi | H | \Psi \rangle}{\langle \Psi | \Psi \rangle} \longrightarrow \min_{A} \frac{\bar{A} H_{\text{eff}} A}{\bar{A} N_{\text{eff}} A}$$

sweep back and forth
over tensors

White, PRL 1992
Verstraete, Porras, Cirac, PRL 2004
Schollwöck, RMP 2005, Ann. Phys. 2011

BASIC ALGORITHMS

variational minimization of energy

$$\min_A \frac{\bar{A} H_{\text{eff}} A}{\bar{A} N_{\text{eff}} A} \longrightarrow H_{\text{eff}} A = \lambda_{\min} N_{\text{eff}} A$$

canonical form
tensor structure
sparse matrices

} $O(D^3)$

DMRG

guaranteed convergence

PBC: higher cost $\mathbf{O}(D^5)$

also excitations

symmetries can be integrated

PEPS $\mathbf{O}(D^{10})$

an example

Ising model $H = J\sum \sigma_z^{[i]}\sigma_z^{[i+1]} + g\sum \sigma_x^{[i]}$ $J=1$ $g=1$

$N=20$ $2^N = 1048576$ $D_{\max} = 2^{N/2} = 1024$

Matlab → eigs $E_{\mathrm{GS}}/N = -1.25538985558119$

t=38 s

MPS

bin — -bash ▸ Emacs-x86_64-10_9 — 114×38

```
Initialized arguments: L=20, J=1, g=1, h=0, outfile=testIsing_L20_J1g1.txt, app=0, D=10
Constructed default Contractor
Initialized Contractor
Initialized random state, norm 1-1.110223025e-16i
Created the Hamiltonian
Constructed the hamil MPO
Initial value, with initial state
1.531982113+1.335737077e-16i
Starting findGroundState with initial value E=1.531982113
0       10      0.07659910564
1       10      -1.255389855
Ground state found for D=10 with eigenvalue -25.10779711 (Energy per particle=-1.255389855)  time=0.590109
Starting findGroundState with initial value E=-25.10779711
0       20      -1.255389855
Ground state found for D=20 with eigenvalue -25.10779711 (Energy per particle=-1.255389856)  time=0.74278
Starting findGroundState with initial value E=-25.10779711
0       30      -1.255389856
Ground state found for D=30 with eigenvalue -25.10779711 (Energy per particle=-1.255389856)  time=1.43884
Starting findGroundState with initial value E=-25.10779711
0       40      -1.255389856
Ground state found for D=40 with eigenvalue -25.10779711 (Energy per particle=-1.255389856)  time=2.700611
Starting findGroundState with initial value E=-25.10779711
0       50      -1.255389856
Ground state found for D=50 with eigenvalue -25.10779711 (Energy per particle=-1.255389856)  time=4.59483
Starting findGroundState with initial value E=-25.10779711
0       60      -1.255389856
Ground state found for D=60 with eigenvalue -25.10779711 (Energy per particle=-1.255389856)  time=6.448116
Starting findGroundState with initial value E=-25.10779711
0       70      -1.255389856
```

an example

Ising model $H = J \sum \sigma_z^{[i]} \sigma_z^{[i+1]} + g \sum \sigma_x^{[i]}$ $\quad J = 1 \quad g = 1$

$N = 20 \quad 2^N = 1048576 \quad D_{\max} = 2^{N/2} = 1024$

$N = 200 \quad 2^N \sim 10^{60} \quad D_{\max} = 2^{N/2} \sim 10^{30}$

MPS

bin — -bash ▸ Emacs-x86_64-10_9 — 112×31

```
Initialized arguments: L=200, J=1, g=1, h=0, outfile=testIsing_L200_J1g1.txt, app=0, D=10
Constructed default Contractor
Initialized Contractor
Initialized random state, norm 1+8.326672685e-17i
Created the Hamiltonian
Constructed the hamil MPO
Initial value, with initial state
10.37197956+7.21644966e-16i
Starting findGroundState with initial value E=10.37197956
0       10      0.05185989779
1       10      -1.271418419
Ground state found for D=10 with eigenvalue -254.2849239 (Energy per particle=-1.271424619)  time=8.0116
Starting findGroundState with initial value E=-254.2849239
0       20      -1.271424619
Ground state found for D=20 with eigenvalue -254.2851697 (Energy per particle=-1.271425849)  time=16.437139
Starting findGroundState with initial value E=-254.2851697
0       30      -1.271425849
Ground state found for D=30 with eigenvalue -254.2851813 (Energy per particle=-1.271425906)  time=42.901364
Starting findGroundState with initial value E=-254.2851813
0       40      -1.271425906
Ground state found for D=40 with eigenvalue -254.2851816 (Energy per particle=-1.271425908)  time=102.71914
Starting findGroundState with initial value E=-254.2851816
0       50      -1.271425908
Ground state found for D=50 with eigenvalue -254.2851816 (Energy per particle=-1.271425908)  time=197.829282
Starting findGroundState with initial value E=-254.2851816
0       60      -1.271425908
Ground state found for D=60 with eigenvalue -254.2851816 (Energy per particle=-1.271425908)  time=298.47571
Starting findGroundState with initial value E=-254.2851816
0       70      -1.271425908
```

BASIC ALGORITHMS

approximate action of local operators

local truncation

variational truncation

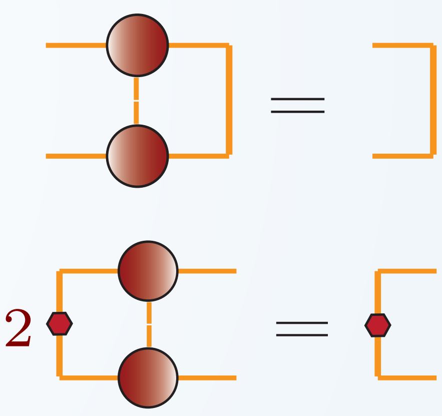
basis of most common evolution algorithms

disclaimer:
also other
algorithms
exist

BASIC ALGORITHMS

local truncation: TEBD

recall canonical form

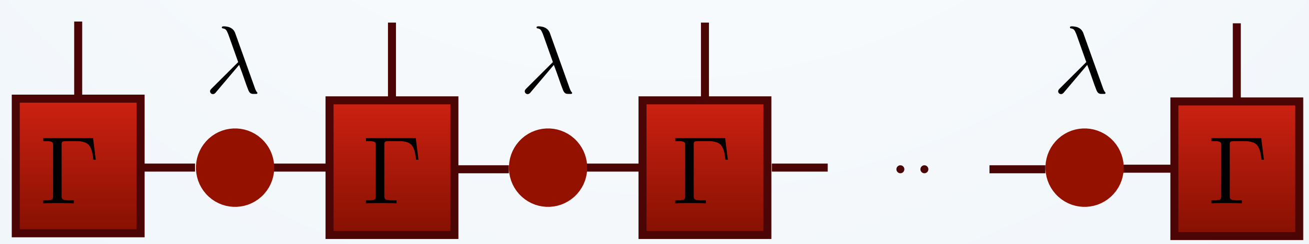


can be made explicit

Schmidt values

$$\Lambda^{[k]} = {\lambda^{[k]}}^2$$

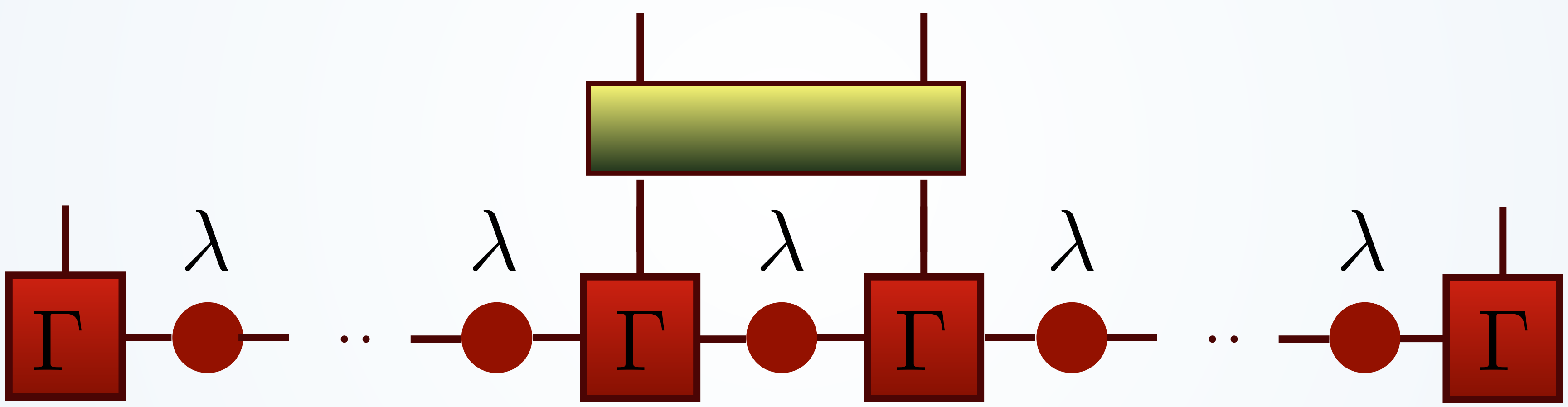
$$A^{[k]} = \Gamma^{[k]} \lambda^{[k]}$$



Vidal, PRL 2003, 2004

BASIC ALGORITHMS

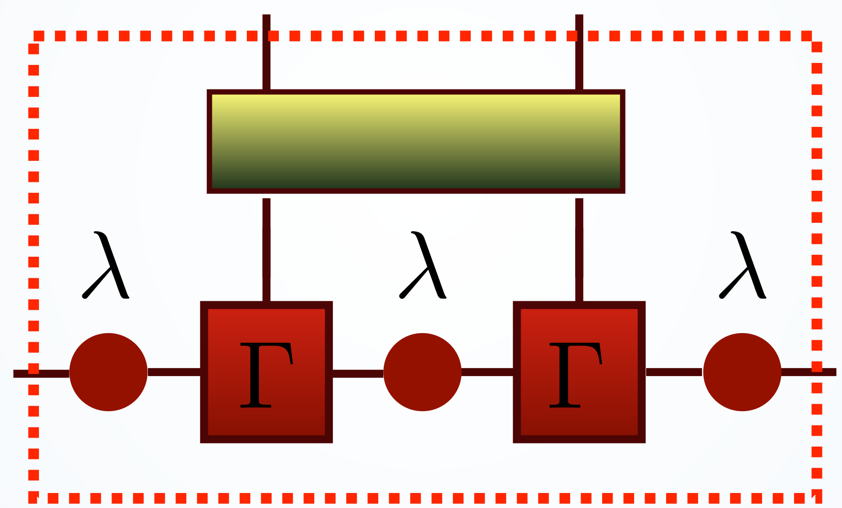
local truncation: TEBD

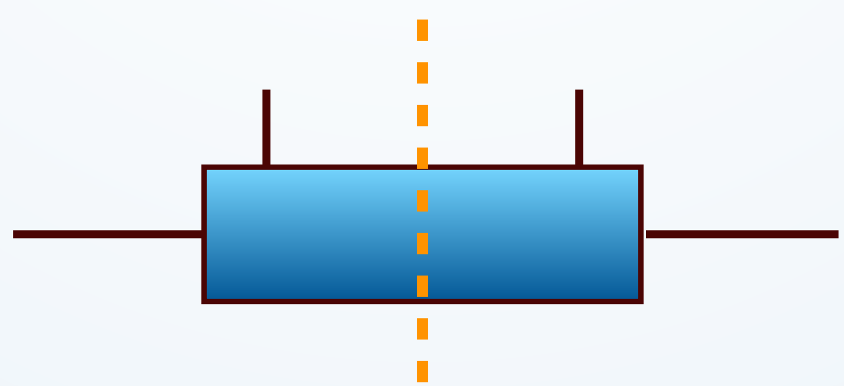


Vidal, PRL 2003, 2004

BASIC ALGORITHMS

local truncation: TEBD

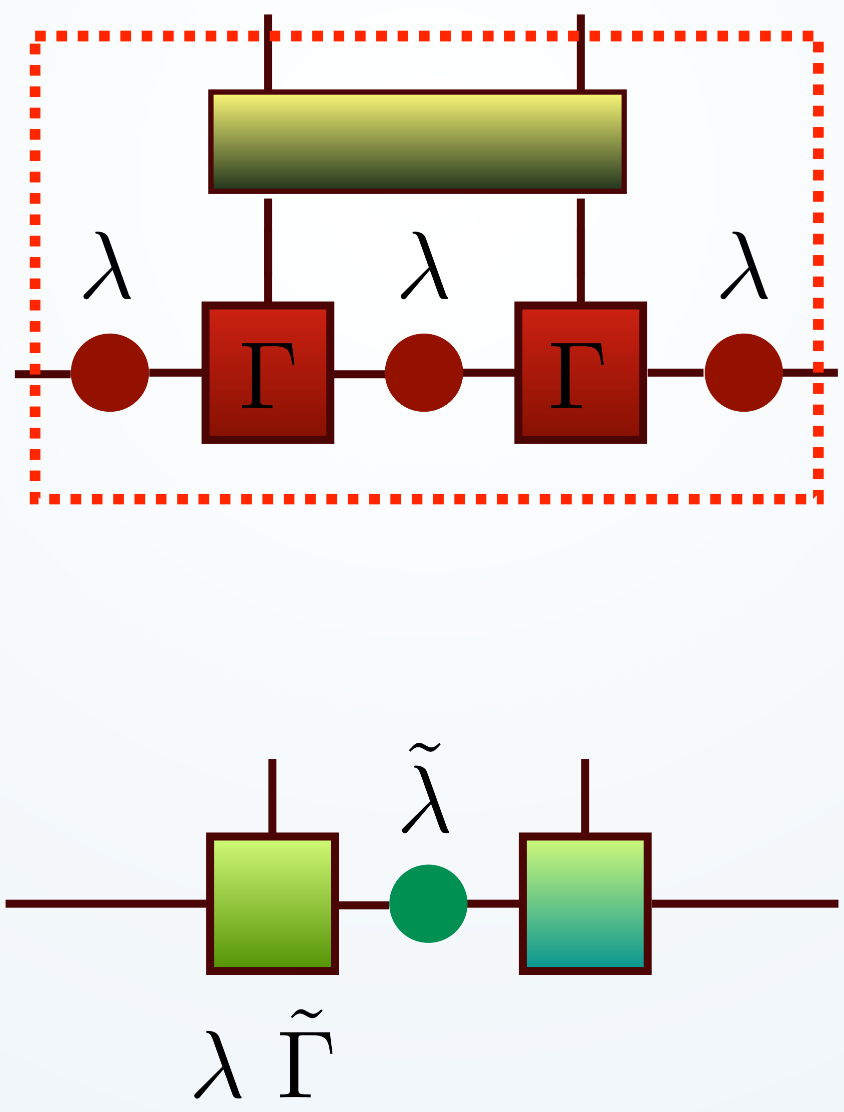




Vidal, PRL 2003, 2004

BASIC ALGORITHMS

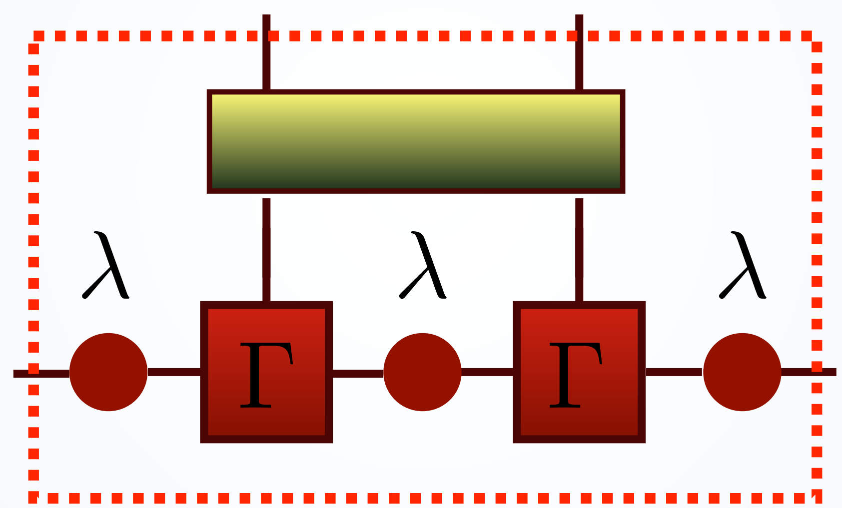
local truncation: TEBD

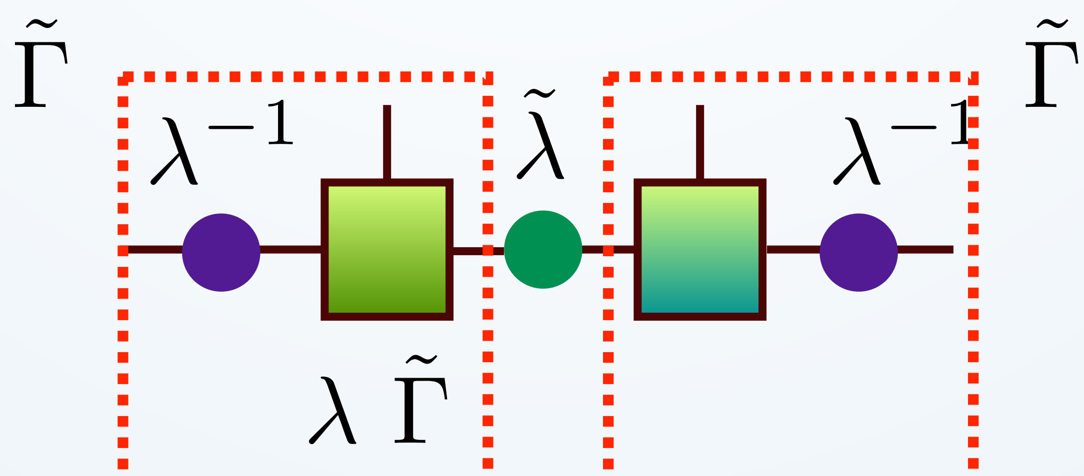


Vidal, PRL 2003, 2004

BASIC ALGORITHMS

local truncation: TEBD





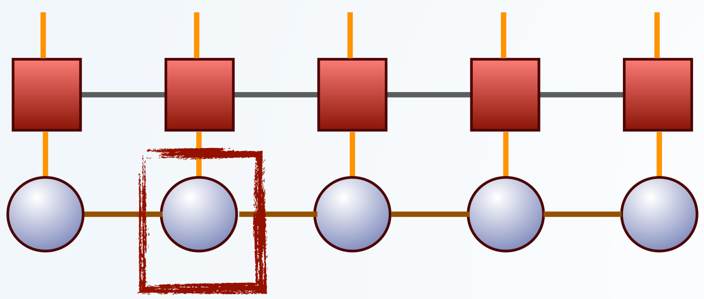
also possible in the TD limit

Vidal, PRL 2003, 2004

BASIC ALGORITHMS

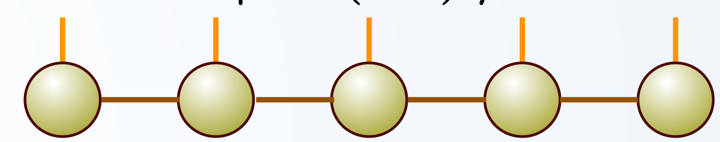
global truncation

$|\Phi_0\rangle$



$\approx$

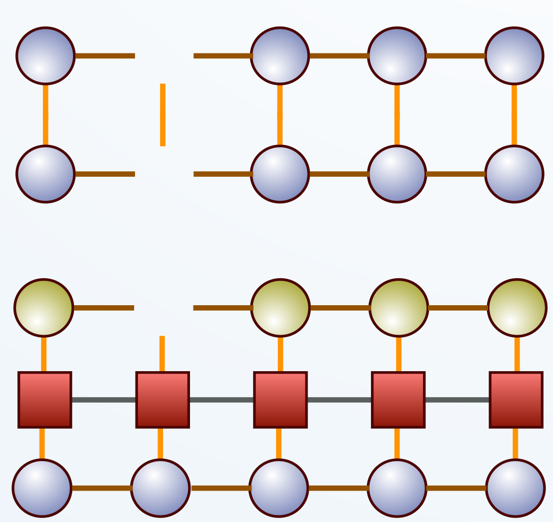
$|\Psi(A)\rangle$



$$\min_{\{A\}} \| |\Psi\rangle - O|\Phi_0\rangle \|^2 \longrightarrow \min_{A}(\bar{A} N_{\text{eff}} A - \bar{A} M - \bar{M} A + \text{const})$$

N_{eff}

M



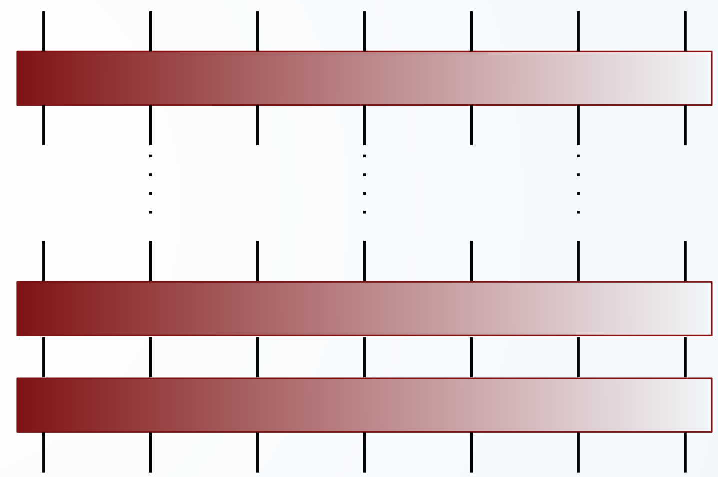
$$N_{\text{eff}} A = M$$

TIME EVOLUTION WITH MPS

BASIC ALGORITHMS

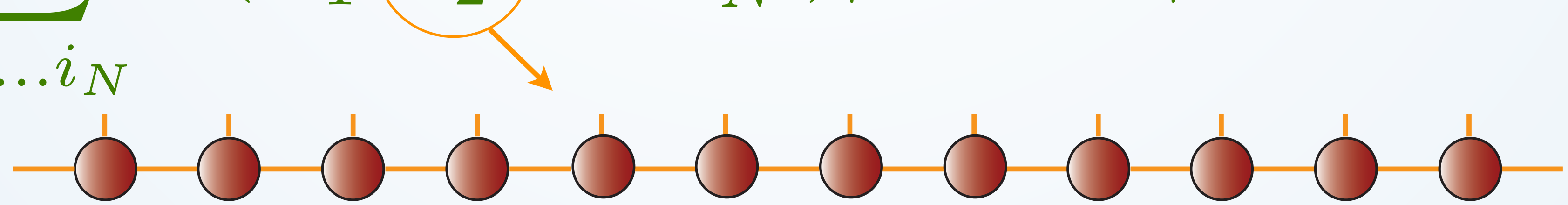
simulate time evolution

$$U(t) \rightarrow [U(\delta)]^M$$



initial MPS

$$|\Psi\rangle = \sum_{i_1 \dots i_N} \mathrm{tr}(A_1^{i_1} A_2^{i_2} \dots A_N^{i_N}) |i_1 \dots i_N\rangle$$



TEBD
t-DMRG

Vidal, PRL 2003, 2004

Verstraete, García-Ripoll, Cirac, PRL 2004

BASIC ALGORITHMS

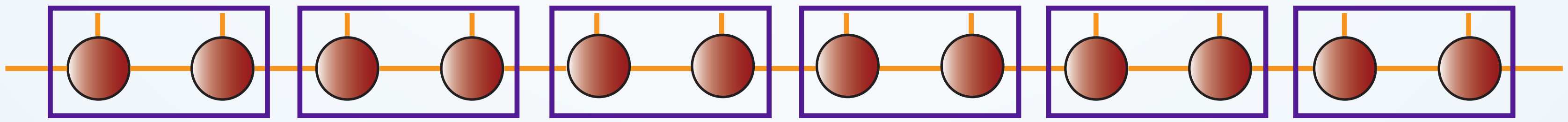
simulate time evolution

$$U(t) \to [U(\delta)]^M$$

apply evolution step

$$H = \boxed{H_e} + H_o$$

Suzuki-Trotter expansion



TEBD
t-DMRG

Vidal, PRL 2003, 2004

Verstraete, García-Ripoll, Cirac, PRL 2004

BASIC ALGORITHMS

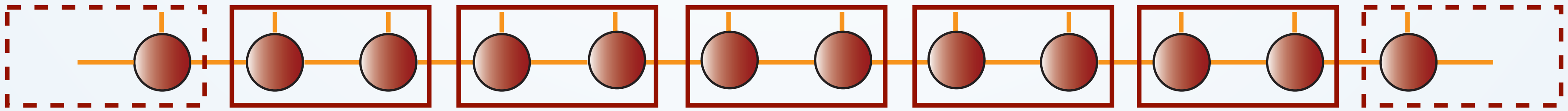
simulate time evolution

$$U(t) \to [U(\delta)]^M$$

$$H = H_e + \boxed{H_o}$$

apply evolution step

Suzuki-Trotter expansion



TEBD
t-DMRG

Vidal, PRL 2003, 2004

Verstraete, García-Ripoll, Cirac, PRL 2004

BASIC ALGORITHMS

simulate time evolution

$$U(t) \to [U(\delta)]^M$$

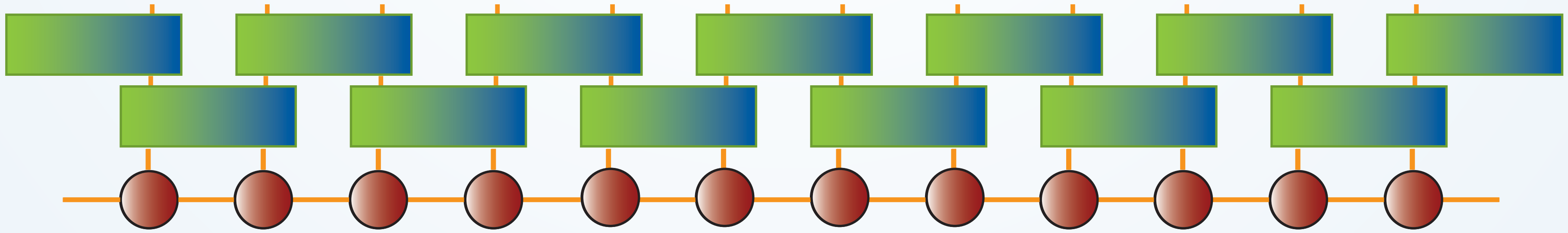
apply evolution step

$$H = H_e + H_o$$

Suzuki-Trotter expansion

$$U(\delta) \approx e^{-iH_e\delta} e^{-iH_o\delta}$$

as local terms!



TEBD
t-DMRG

Vidal, PRL 2003, 2004

Verstraete, García-Ripoll, Cirac, PRL 2004

BASIC ALGORITHMS

simulate time evolution

$$U(t) \rightarrow [U(\delta)]^M$$

apply evolution step

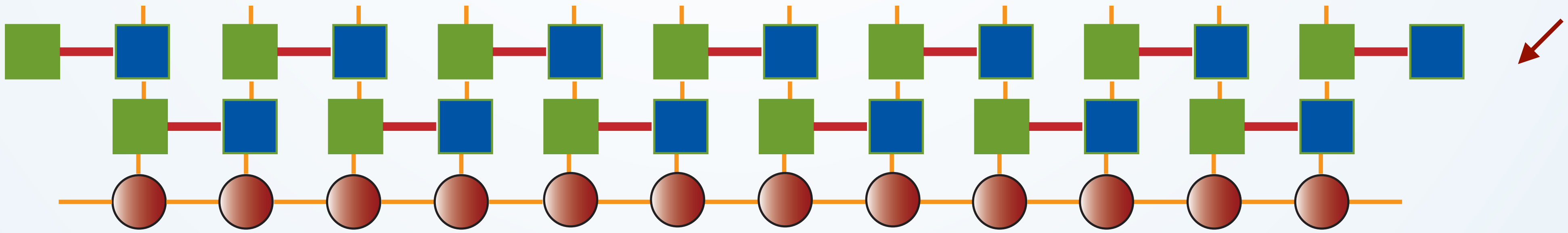
$$H = H_e + H_o$$

Suzuki-Trotter expansion

$$U(\delta) \approx e^{-iH_e\delta} e^{-iH_o\delta}$$

as local terms!

MPOs



TEBD
t-DMRG

Vidal, PRL 2003, 2004

Verstraete, García-Ripoll, Cirac, PRL 2004

basic time evolution algorithms

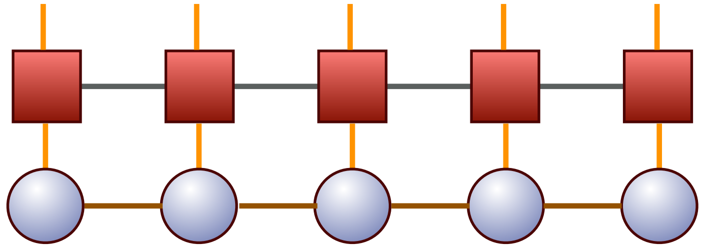
initial MPS

discrete time

$$U(t) \to [U(\delta)]^M$$

Suzuki-Trotter expansion

$$U(\delta) \approx e^{-iH_e\delta} e^{-iH_o\delta}$$

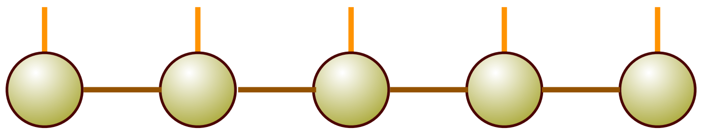


TEBD, t-DMRG

Vidal, PRL 2003, 2004

Verstraete, García-Ripoll, Cirac, PRL 2004

basic time evolution algorithms



initial MPS

discrete time

$$U(t) \to [U(\delta)]^M$$

Suzuki-Trotter expansion

$$U(\delta) \approx e^{-iH_e\delta} e^{-iH_o\delta}$$

truncate bond dimension

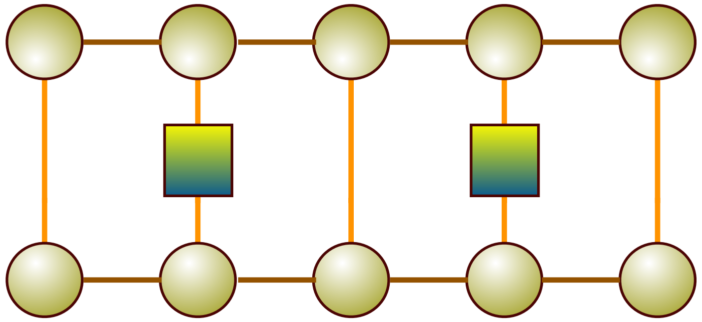
TEBD, t-DMRG

Vidal, PRL 2003, 2004

Verstraete, García-Ripoll, Cirac, PRL 2004

basic time evolution algorithms

time evolved state approximated by MPS



TEBD, t-DMRG

Vidal, PRL 2003, 2004

Verstraete, García-Ripoll, Cirac, PRL 2004

initial MPS

discrete time

$$U(t) \to [U(\delta)]^M$$

Suzuki-Trotter expansion

$$U(\delta) \approx e^{-iH_e\delta} e^{-iH_o\delta}$$

truncate bond dimension

iterate

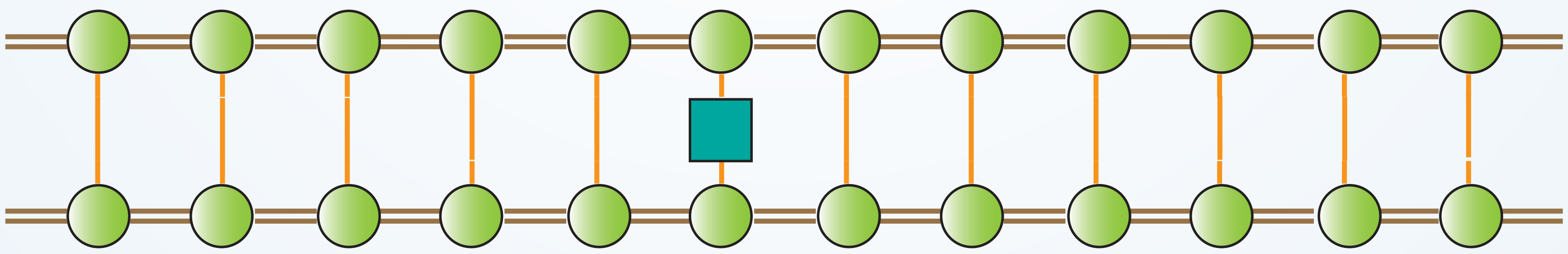
compute observables

BASIC ALGORITHMS

simulate time evolution

works for real and imaginary time

imaginary time for ground states,
thermal equilibrium



compute observables

TEBD
t-DMRG

Vidal, PRL 2003, 2004

Verstraete, García-Ripoll, Cirac, PRL 2004

BASIC ALGORITHMS

imaginary time evolution ⇒ ground state

$$\lim_{\tau\to\infty} e^{-\tau H}|\Phi_0\rangle \to |E_{\min}\rangle$$

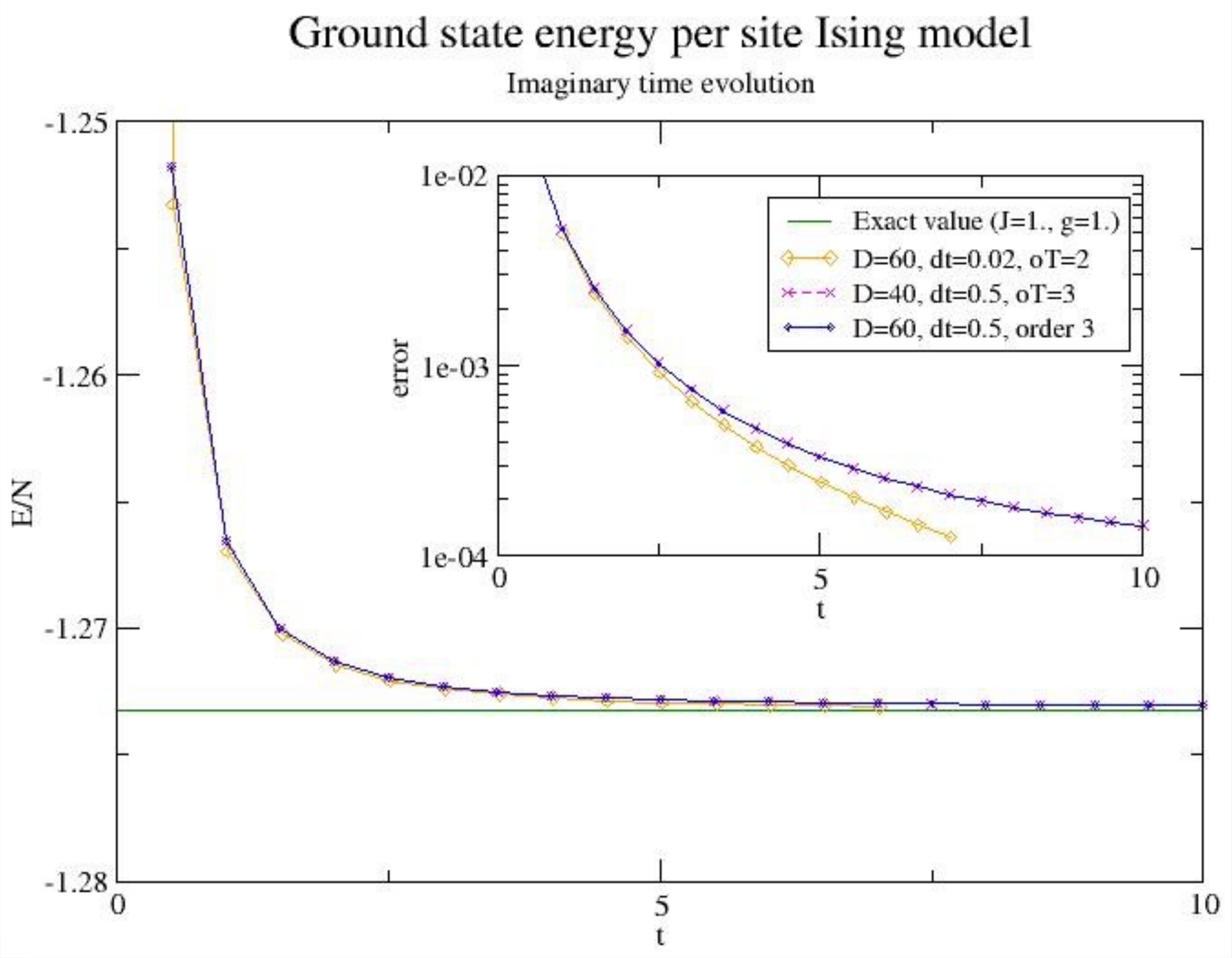
$$|\Phi_0\rangle = \sum c_n|E_n\rangle$$

$$e^{-\tau H}|\Phi_0\rangle = \sum c_n e^{-\tau E_n}|E_n\rangle$$

$$e^{-\tau H}|\Phi_0\rangle \propto c_0|E_0\rangle + \sum_{n>0} c_n e^{-\tau(E_n-E_0)}|E_n\rangle$$

BASIC ALGORITHMS

imaginary time evolution ⇒ ground state



BASIC ALGORITHMS

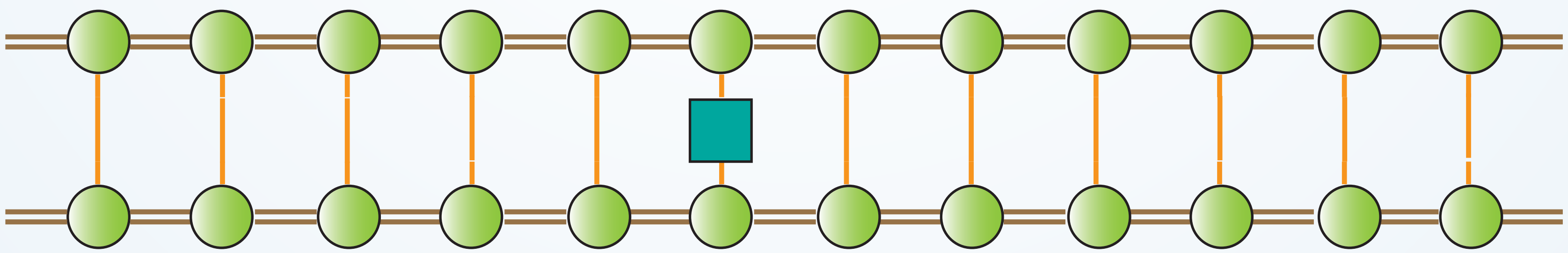
simulate time evolution

works for real and imaginary time

imaginary time for ground states,
thermal equilibrium

but out of equilibrium entanglement can grow fast!

Osborne, PRL 2006
Schuch et al., NJP 2008

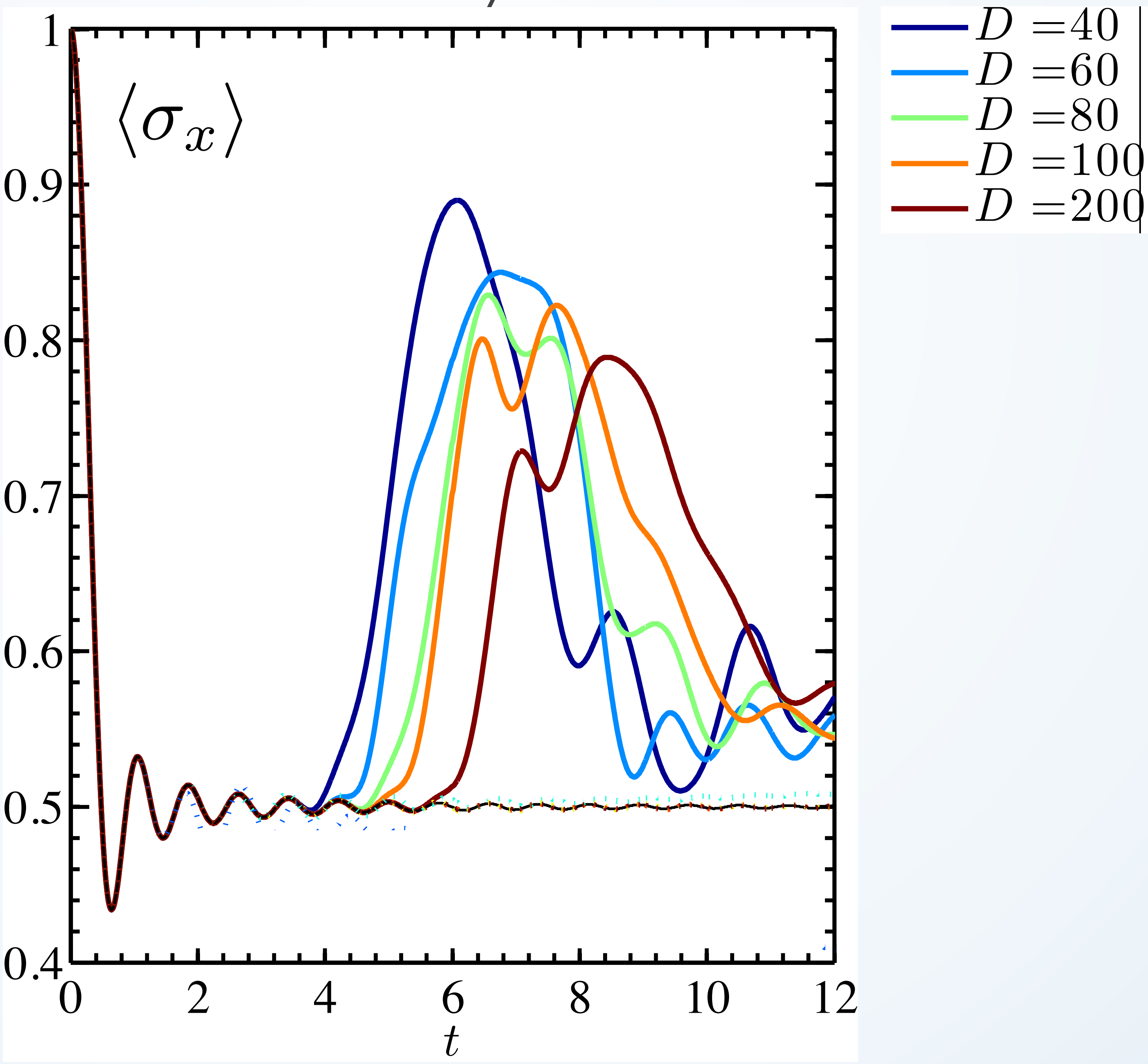


compute
observables

TEBD
t-DMRG

BASIC ALGORITHMS

real time dynamics

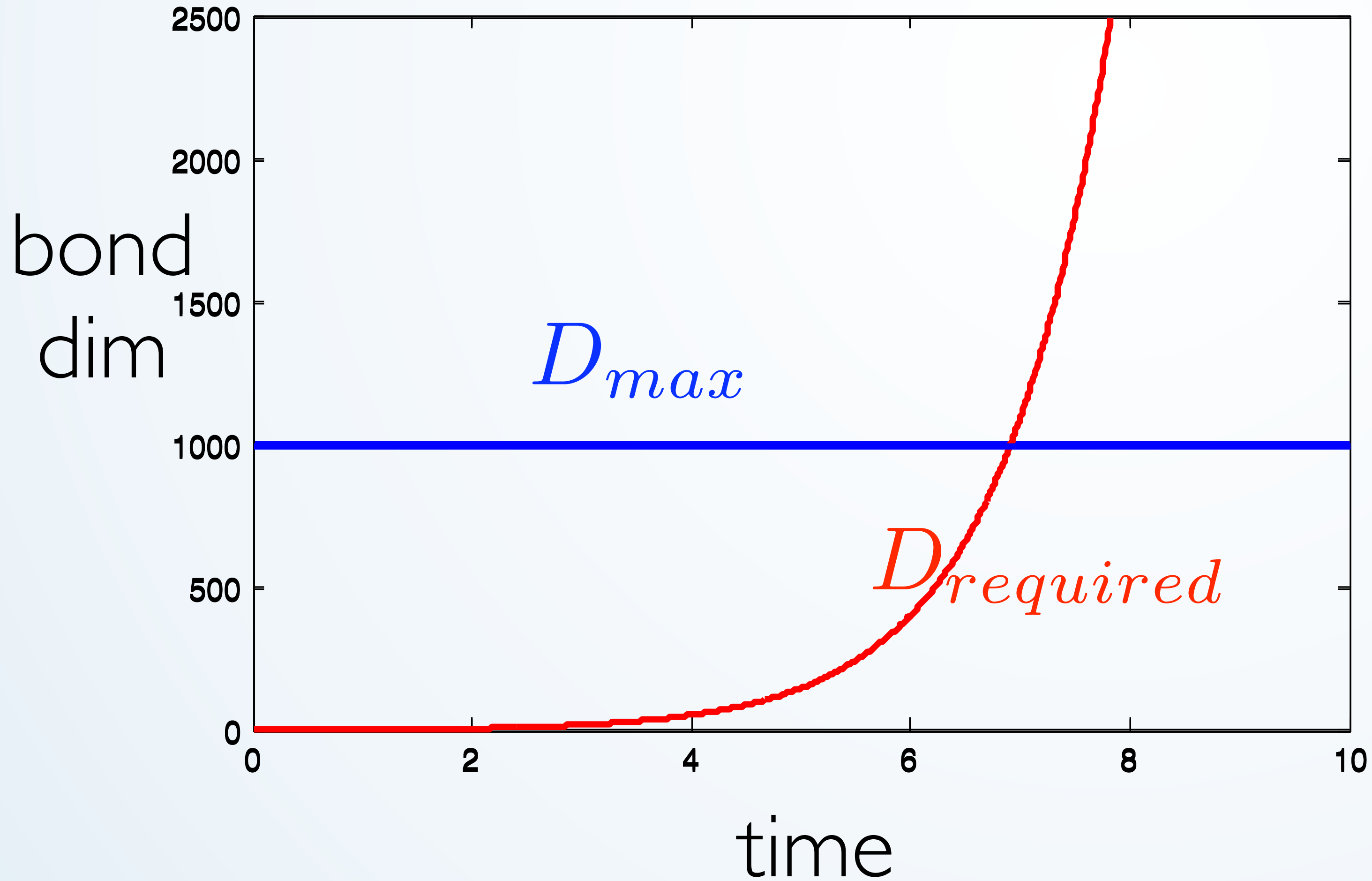


ENTANGLEMENT AND TIME EVOLUTION

ENTANGLEMENT GROWTH

Entropy of evolved state may grow linearly

Osborne, PRL 2006
Schuch et al., NJP 2008



required bond for fixed precision

$$D \sim e^{\alpha t}$$

limits the simulation of out of equilibrium many physical situations (in closed and open quantum systems) can be successfully studied!

short times, adiabatic, low energy can work well

García-Ripoll, NJP 2006
Wall, Carr NJP 2012
Paeckel et al Ann. Phys. 2019

SOME OTHER APPLICATIONS

MIXED STATES

open systems, thermal…

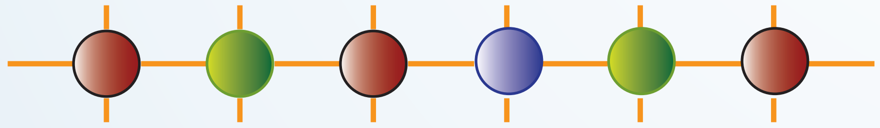
MIXED STATES

similar problems…

equilibrium → thermal states

imaginary time evolution

time-dependent → real time evolution



unitary $\rho(t) = U(t)\rho(0)U(t)^\dagger$

non-unitary $$\frac{d\rho(t)}{dt} = \mathcal{L}(\rho)$$

Verstraete et al., PRL 2004; Prosen, Znidaric PRL 2008
Cai, Barthel, PRL 2013, …

THERMAL STATES

THERMAL STATES

Gibbs ensemble as imaginary time evolution

$$\rho_\beta = \frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})}$$

goal: approximate as MPO

efficient!

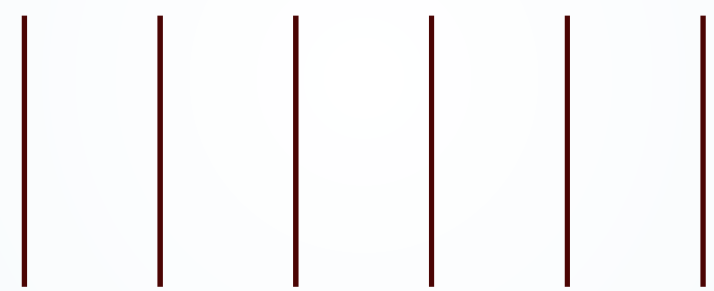
Hastings PRB 2006
Molnar et al PRB 2015

use Suzuki-Trotter expansion for exponential

$$e^{-\beta H} = e^{-\beta H}\mathbb{1} = \left(e^{-\frac{\beta}{M}H}\right)^M \mathbb{1}$$

THERMAL STATES

purification natural

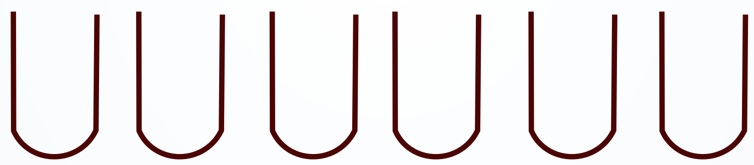


vectorize

$$e^{-\beta H} = e^{-\beta H}\mathbb{1} = \left(e^{-\frac{\beta}{M}H}\right)^M \mathbb{1}$$

THERMAL STATES

purification natural

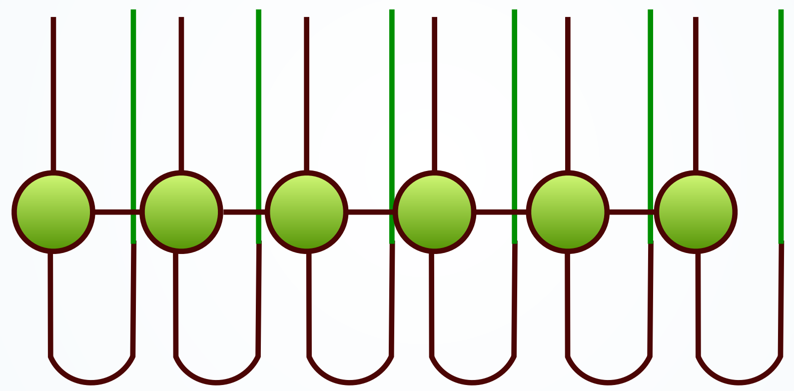


vectorize

$$e^{-\beta H} = e^{-\beta H}\mathbb{1} = \left(e^{-\frac{\beta}{M}H}\right)^M \mathbb{1}$$

THERMAL STATES

purification natural



vectorize

$$e^{-\beta H} = e^{-\beta H}\mathbb{1} = \left(e^{-\frac{\beta}{M}H}\right)^{M}\mathbb{1}$$

THERMAL STATES

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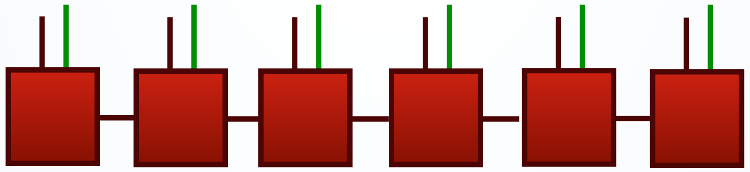
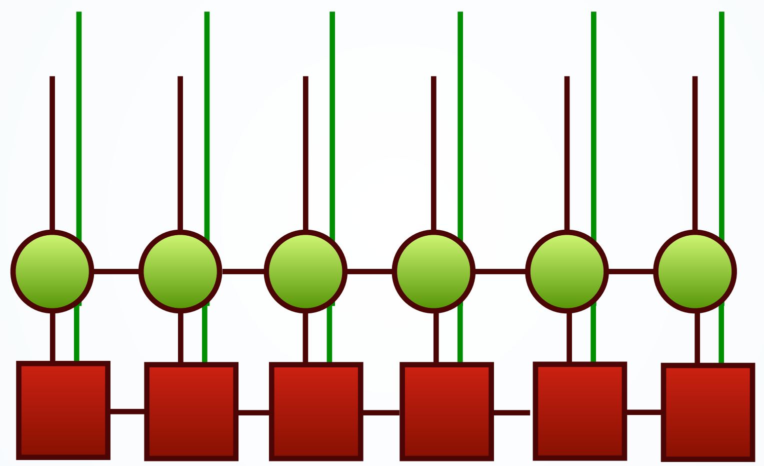
vectorize



$$e^{-\beta H} = e^{-\beta H}\mathbb{1} = \left(e^{-\frac{\beta}{M}H}\right)^M \mathbb{1}$$

THERMAL STATES

purification natural

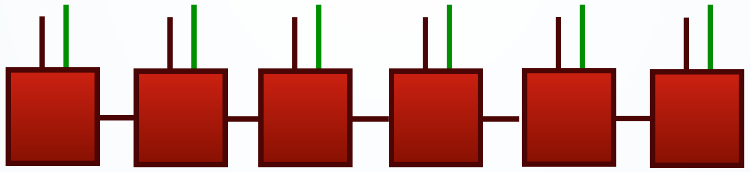


vectorize

$$e^{-\beta H} = e^{-\beta H}\mathbb{1} = \left(e^{-\frac{\beta}{M}H}\right)^{M}\mathbb{1}$$

THERMAL STATES

purification natural

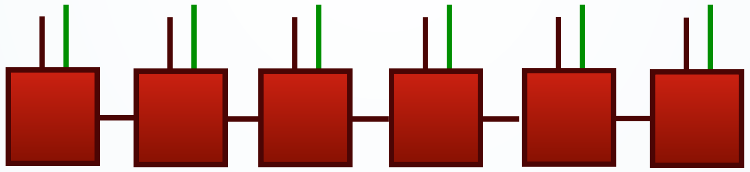


vectorize

THERMAL STATES

purification natural

$$\rho(\beta) \propto \rho(\beta/2)\rho(\beta/2)^\dagger$$



vectorize

THERMAL STATES

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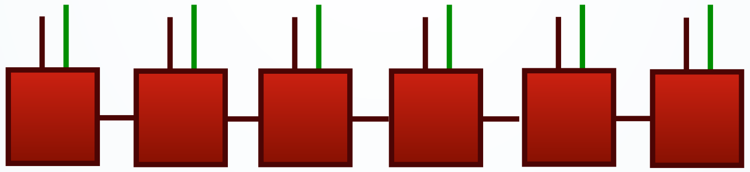


vectorize

combining real time evolution can compute thermal response functions

Barthel, NJP 2013

alternative method: METTS White, PRL 2009

Binder, Barthel, PRB 2015

OPEN SYSTEMS

OPEN SYSTEMS

non-unitary dynamics

Real-time evolution produces a steady state

$$\frac{d\rho(t)}{dt} = \mathcal{L}(\rho) \longrightarrow \mathcal{L}(\rho_S) = 0$$

fixed point of Liouvillian map

dissipative QC
dissipative QPT

We can approximate it as a MPO

simulating long time evolution
~imaginary time evolution

García-Ripoll et al., Zwolak, Prosen, ...

variationally
~DMRG

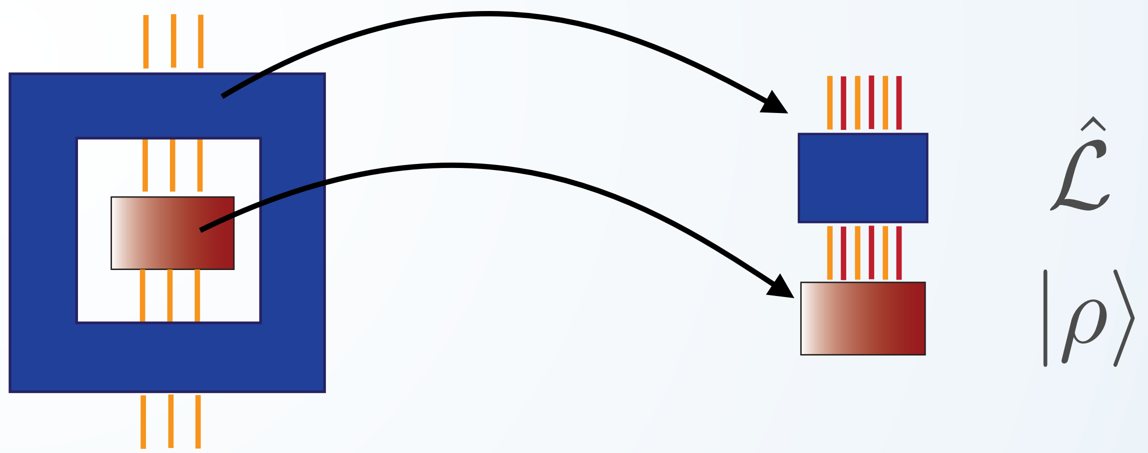
Cui et al., PRL114, (2015)
Mascarenhas et al., PRA92 (2015)

OPEN SYSTEMS

non-unitary dynamics

Dynamics determined by Liouvillian

$$\frac{d\rho}{dt} = \mathcal{L}(\rho)$$



vectorize $|\rho\rangle$

superoperator $\hat{\mathcal{L}}$

$$|\rho(t)\rangle = e^{\mathcal{L}t}|\rho(0)\rangle$$

NESS

fixed point of evolution

$$\hat{\mathcal{L}}|\rho\rangle = 0$$

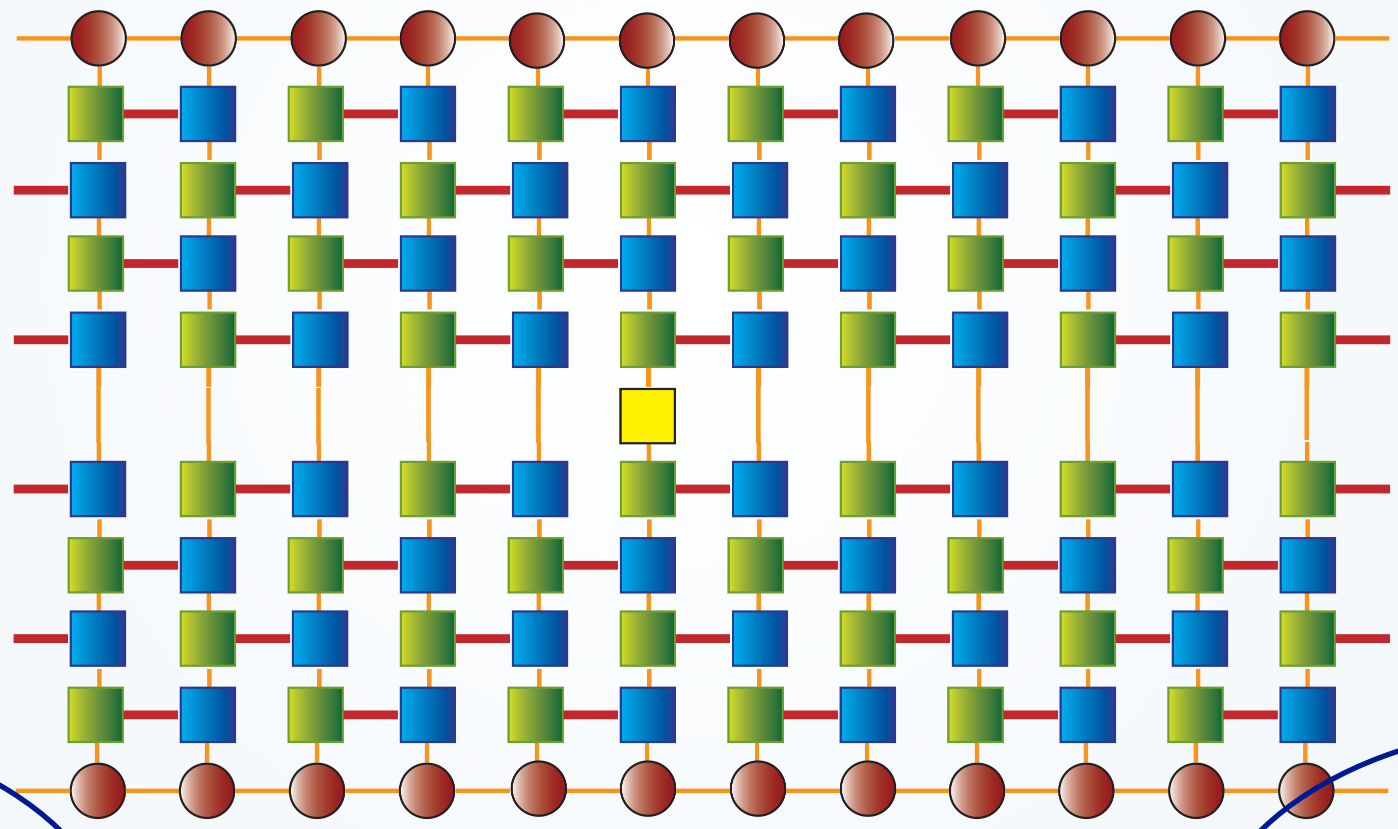
SOME ALTERNATIVES FOR REAL TIME EVOLUTION

avoid representing the state as MPS

ALTERNATIVELY...

time dependent observables as TN

problem is contracting the network



TN describe observables, not states

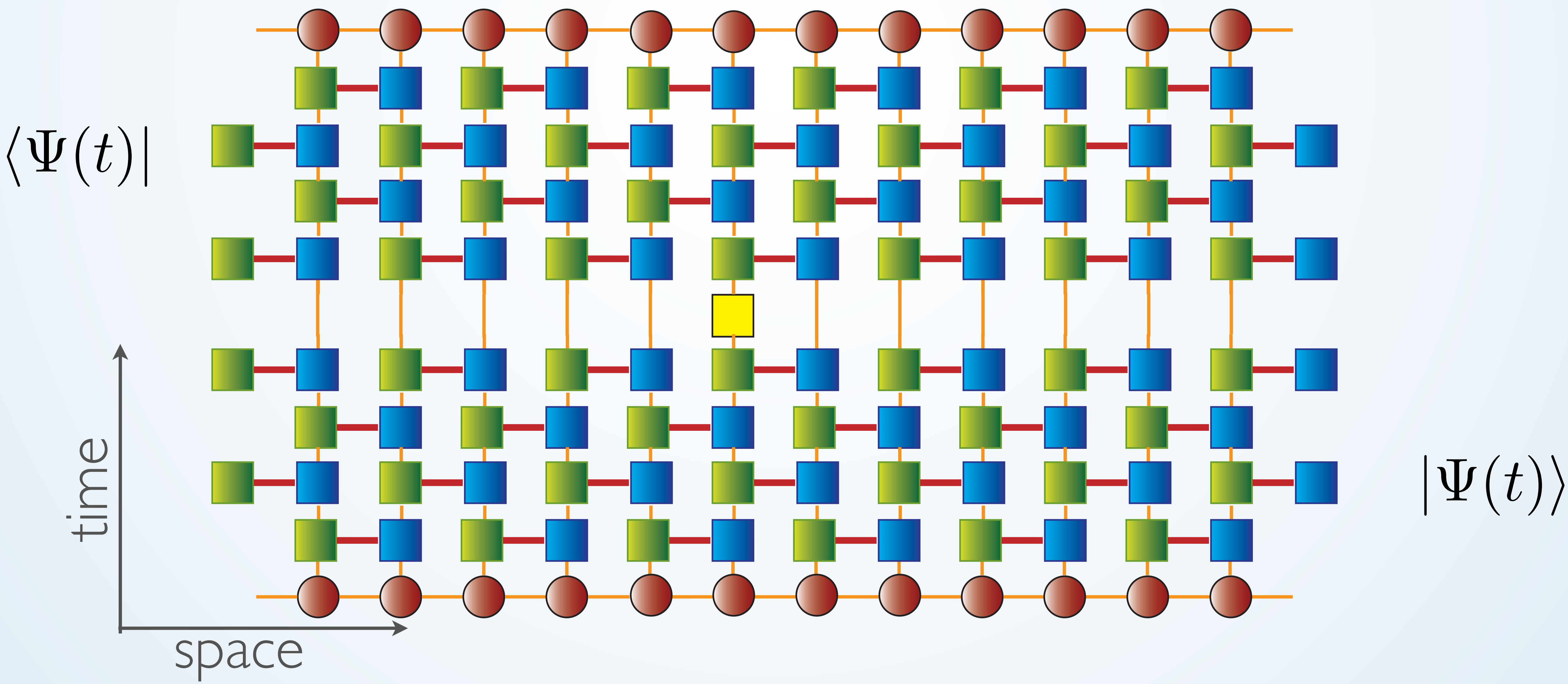
exact contraction not possible

#P complete

key: *entanglement* in TN

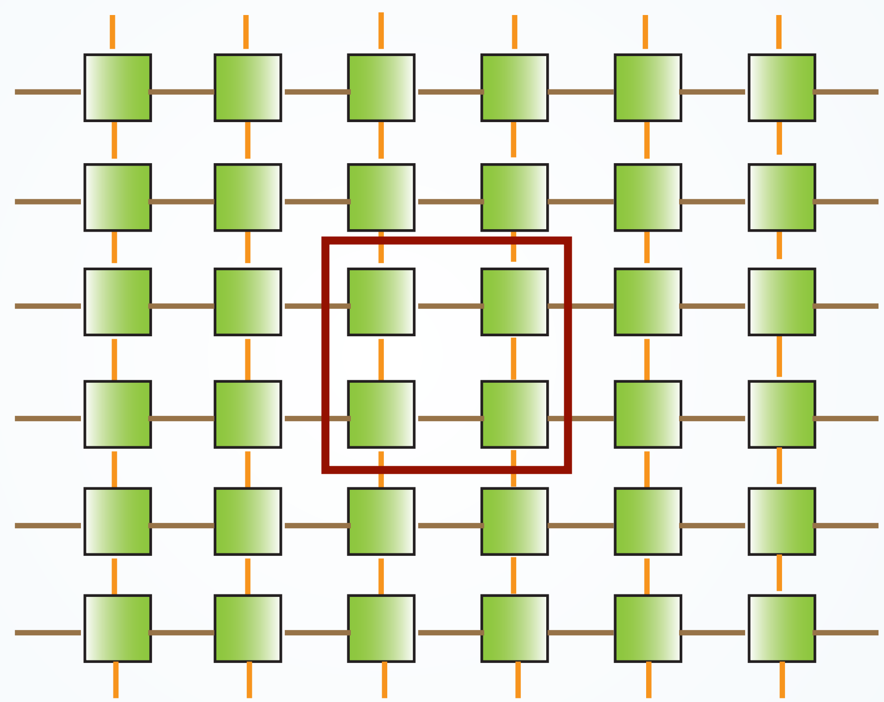
OBSERVABLE AS TN

$$\langle\Psi(t)|O|\Psi(t)\rangle$$



tensor networks may describe partition functions (observables)

need to contract a TN

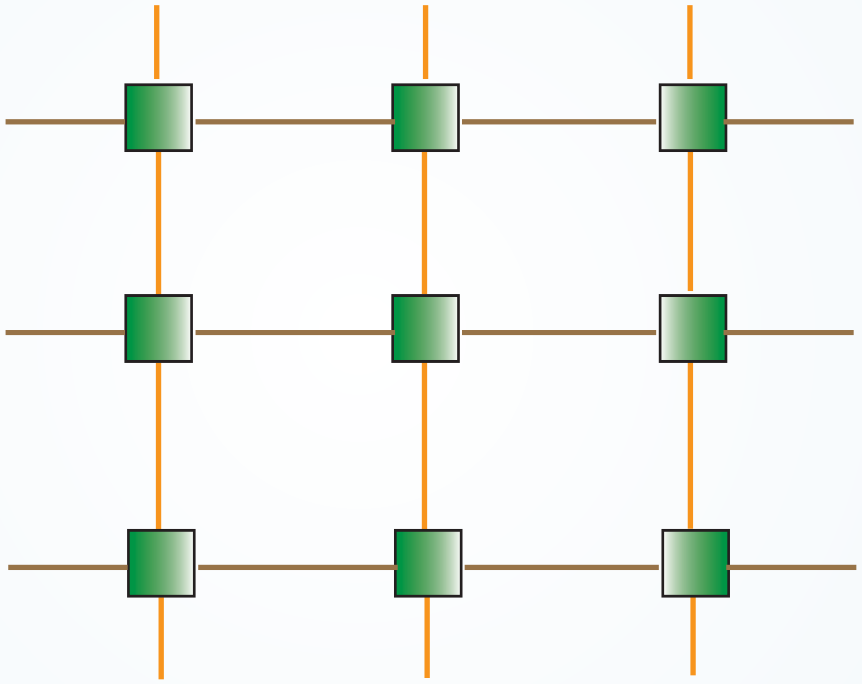


TRG approaches

Nishino, JPSJ 1995
Levin & Wen PRL 2008
Xie et al PRL2009; Zhao et al PRB 2010

tensor networks may describe partition functions (observables)

need to contract a TN

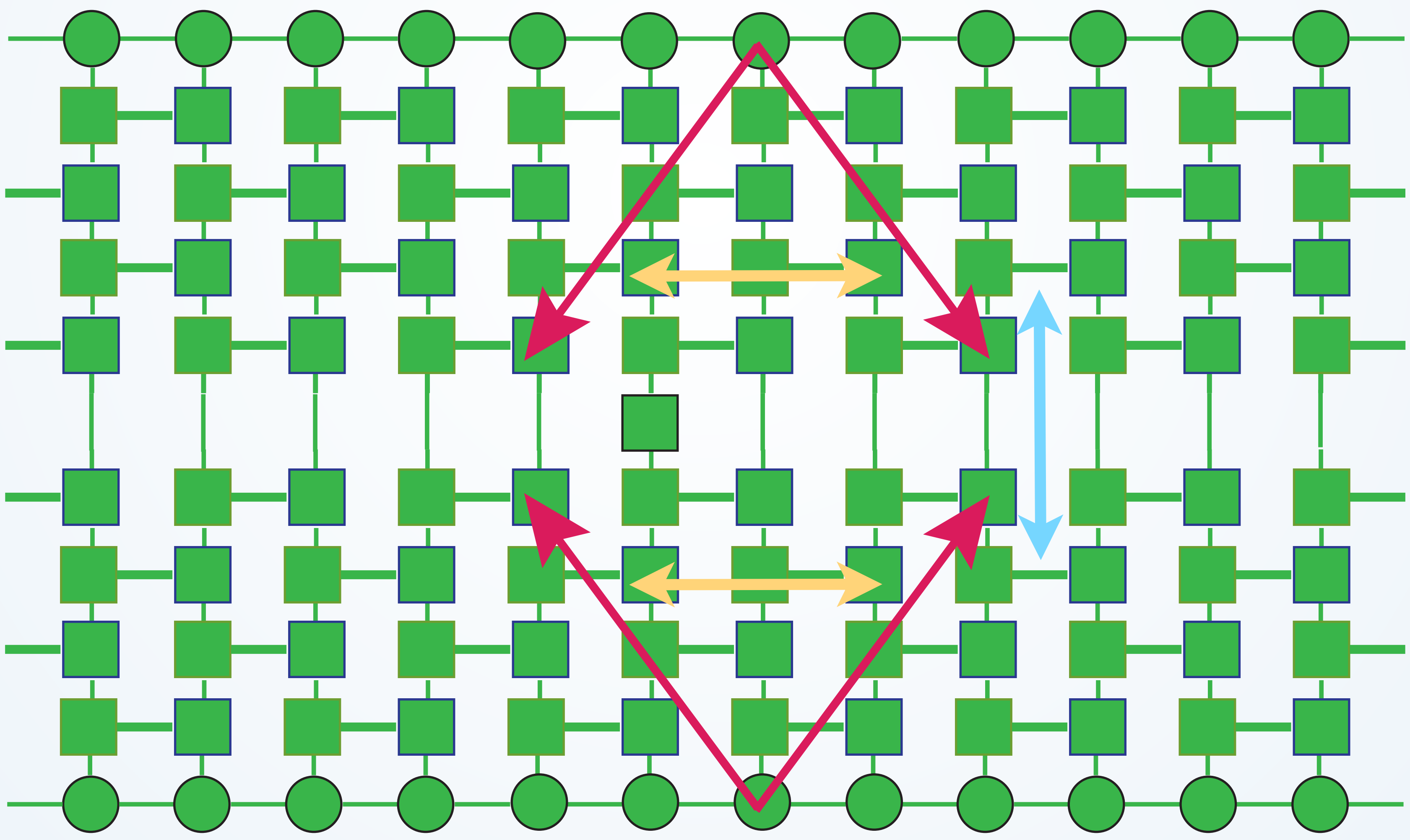


TRG approaches

Nishino, JPSJ 1995
Levin & Wen PRL 2008
Xie et al PRL2009; Zhao et al PRB 2010

OBSERVABLE AS TN

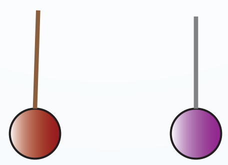
intuition: model free propagating excitations



MCB, Hastings, Verstraete, Cirac, PRL 2009
Müller-Hermes, Cirac, MCB, NJP 2012

Toy Tensor Network model helps to understand entanglement in the network

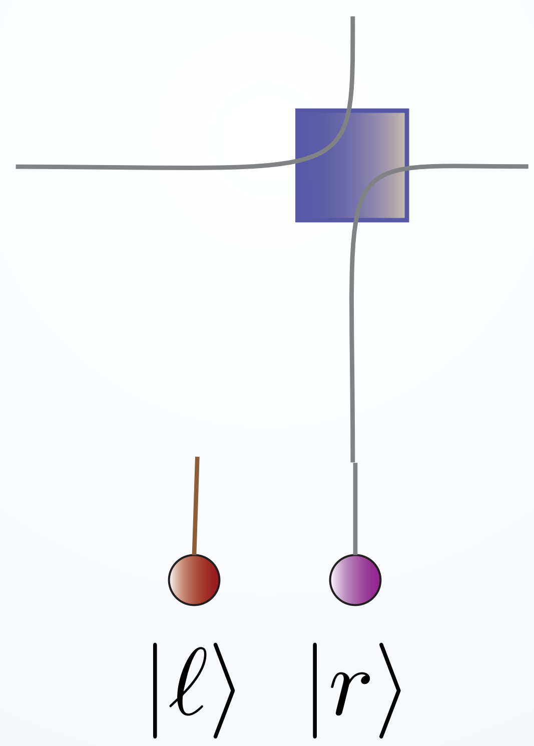
TOY MODEL TN



$|\ell\rangle$ $|r\rangle$

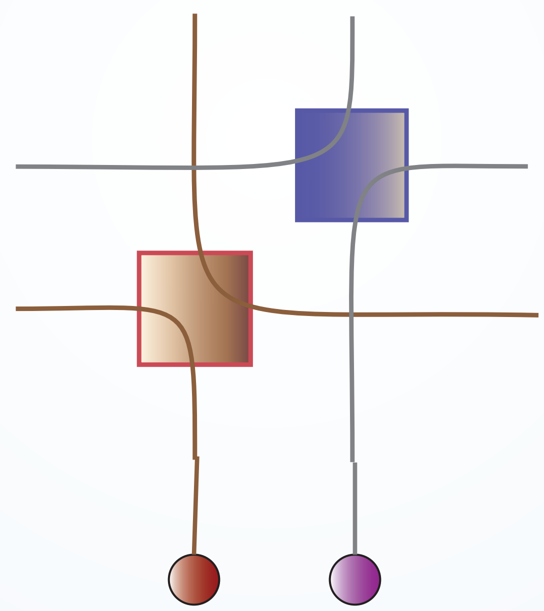
MCB, Hastings, Verstraete, Cirac, PRL 2009
Müller-Hermes, Cirac, MCB, NJP 2012

TOY MODEL TN



MCB, Hastings, Verstraete, Cirac, PRL 2009
Müller-Hermes, Cirac, MCB, NJP 2012

TOY MODEL TN



$|\ell\rangle$ $|r\rangle$

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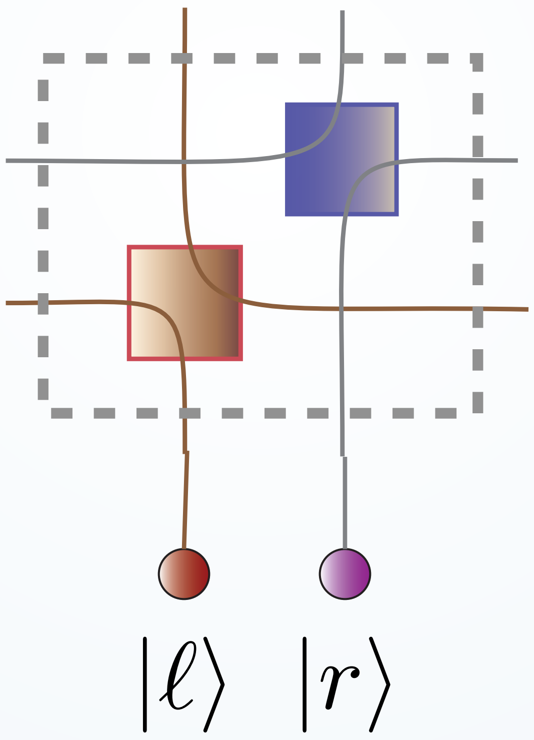
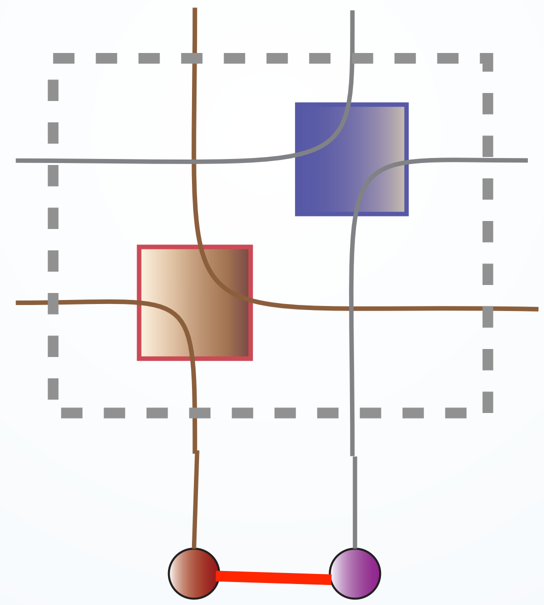
TOY MODEL TN



MCB, Hastings, Verstraete, Cirac, PRL 2009
Müller-Hermes, Cirac, MCB, NJP 2012

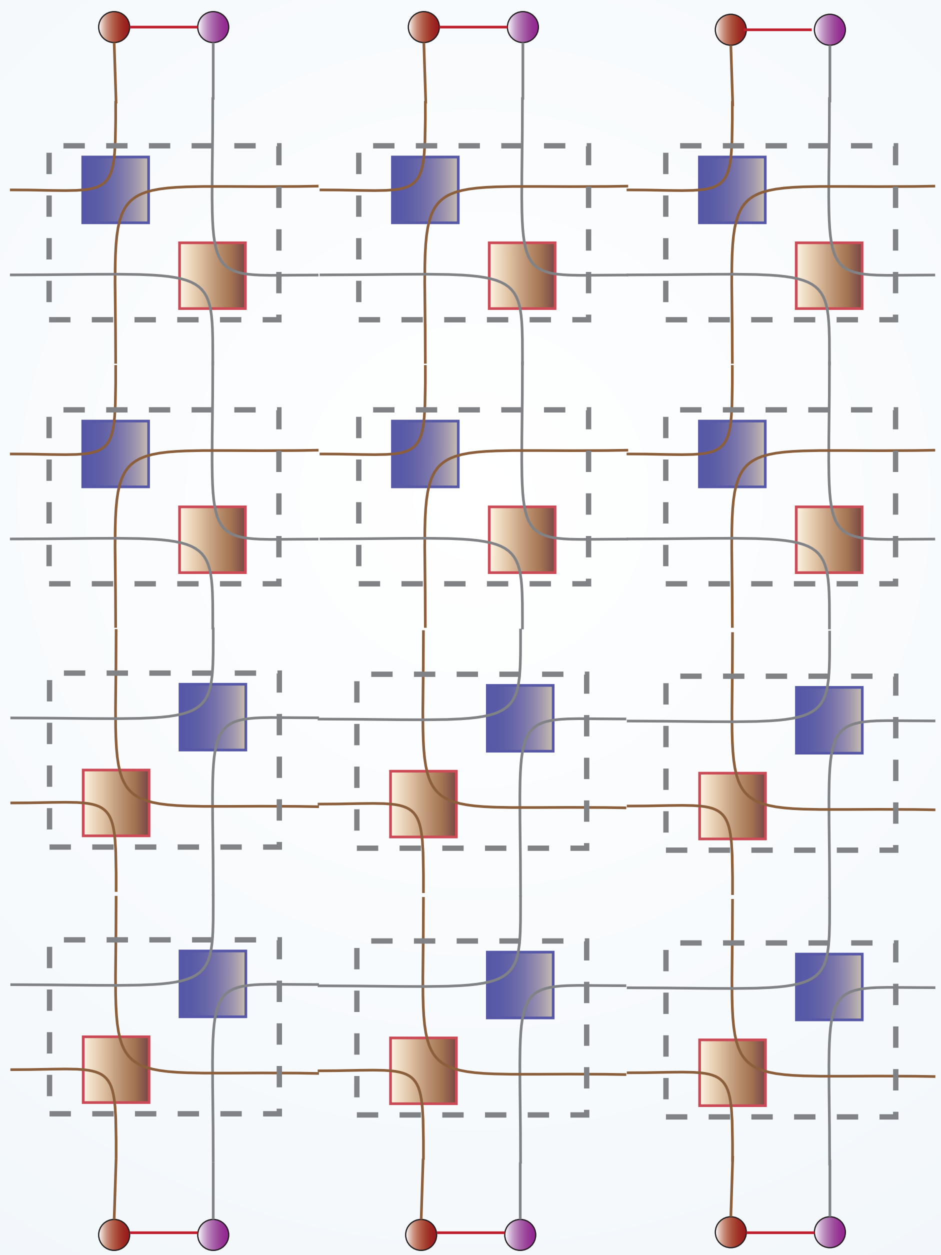
TOY MODEL TN



$$|\ell\rangle \otimes |0\rangle + |0\rangle \otimes |r\rangle$$

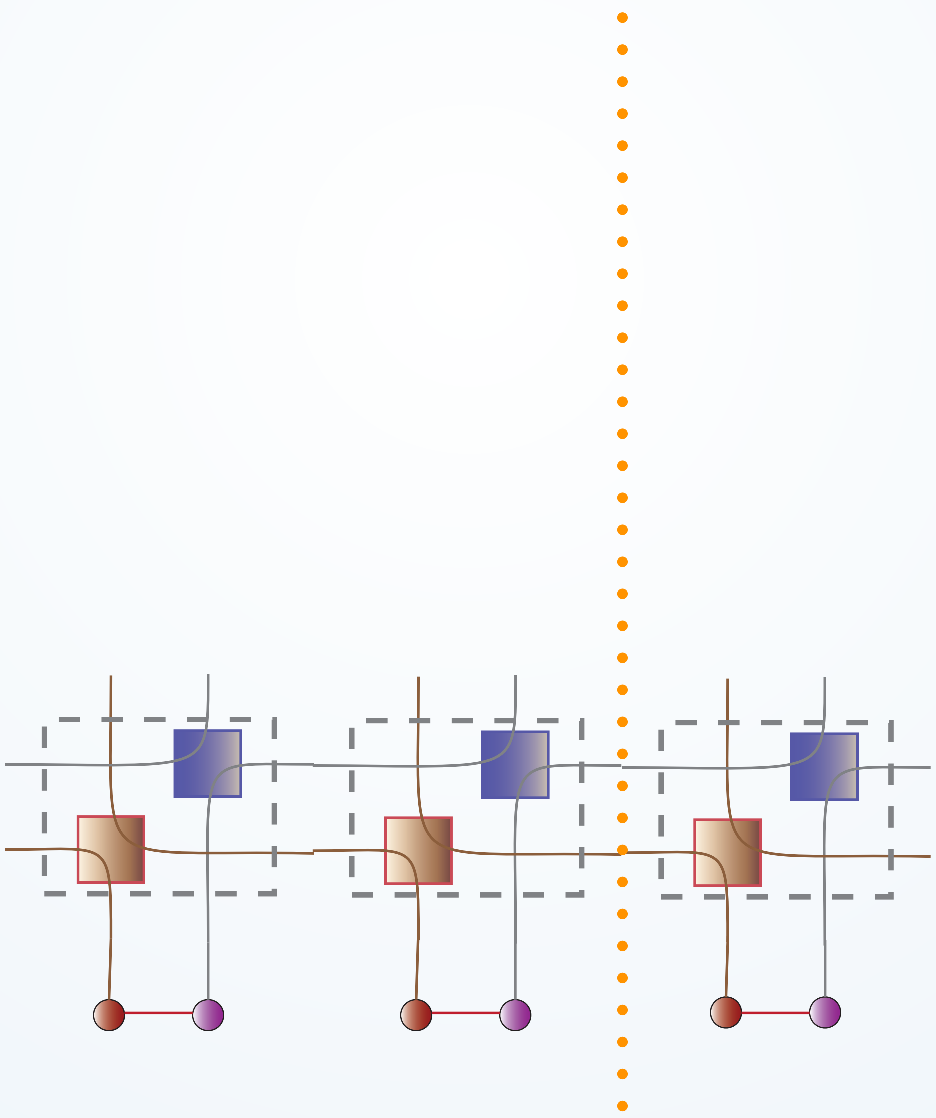
MCB, Hastings, Verstraete, Cirac, PRL 2009
Müller-Hermes, Cirac, MCB, NJP 2012

TOY MODEL TN



MCB, Hastings, Verstraete, Cirac, PRL 2009
Müller-Hermes, Cirac, MCB, NJP 2012

TOY MODEL TN



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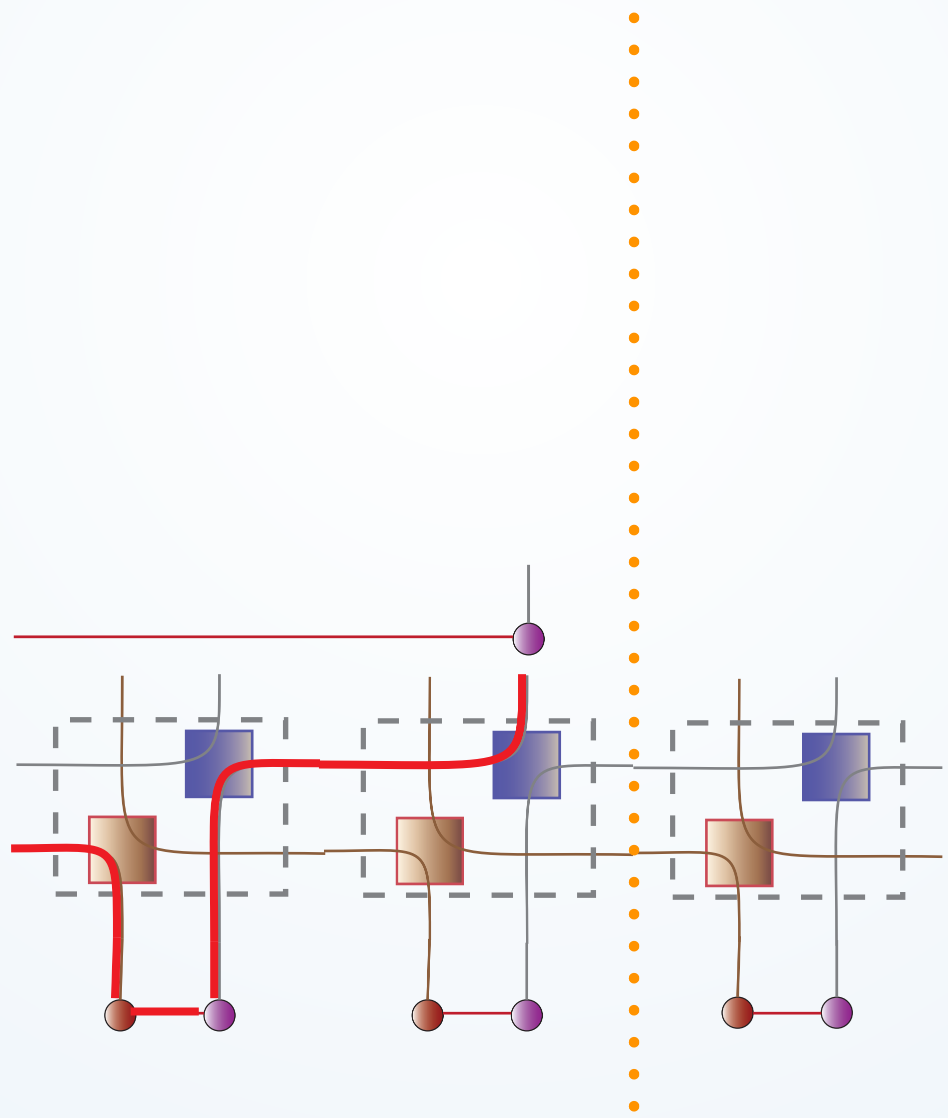

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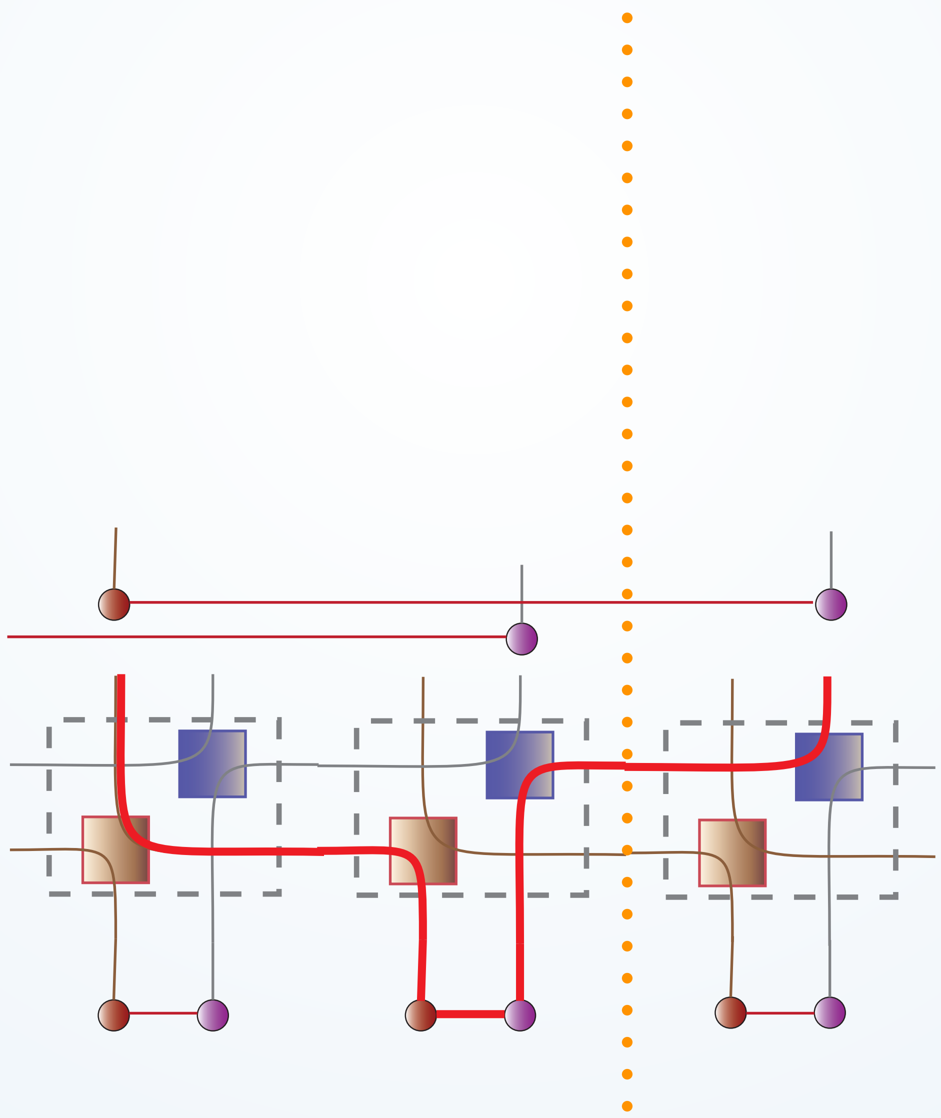

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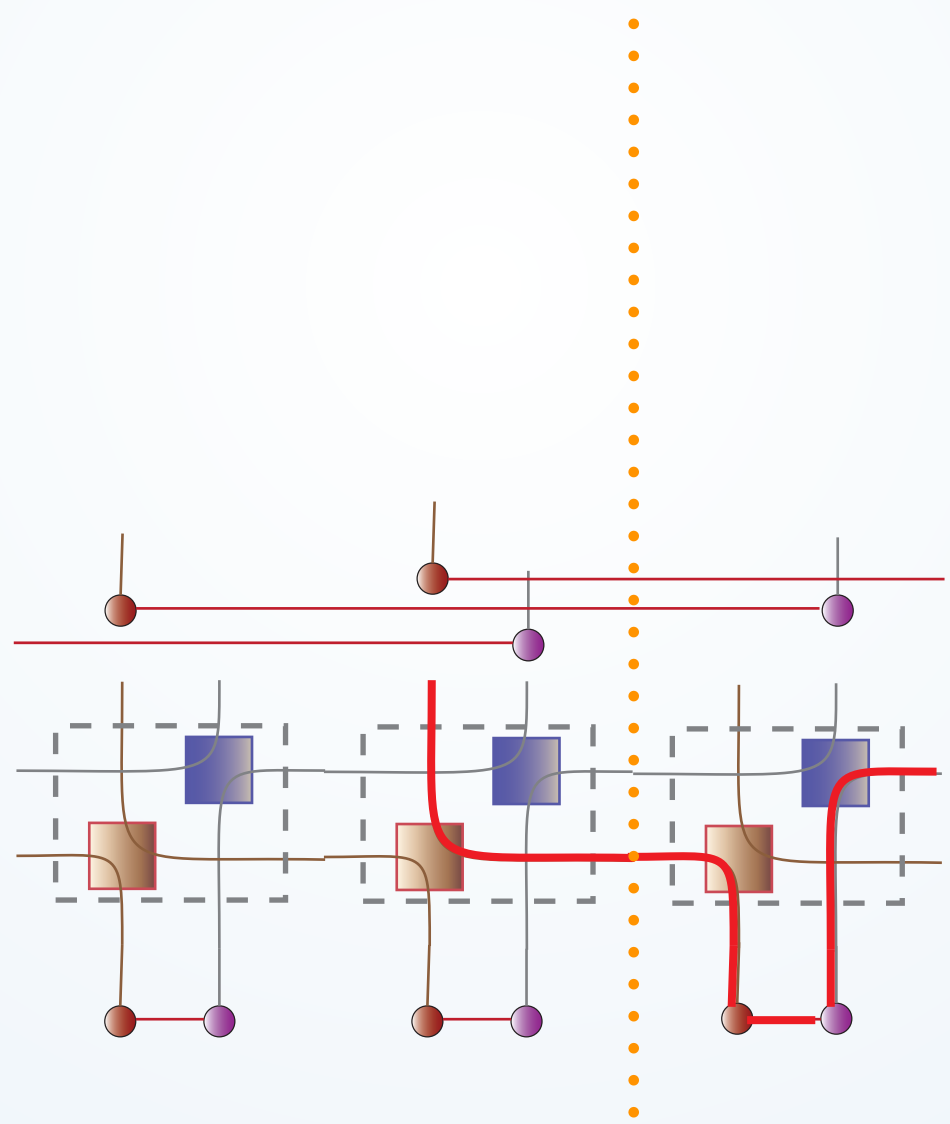

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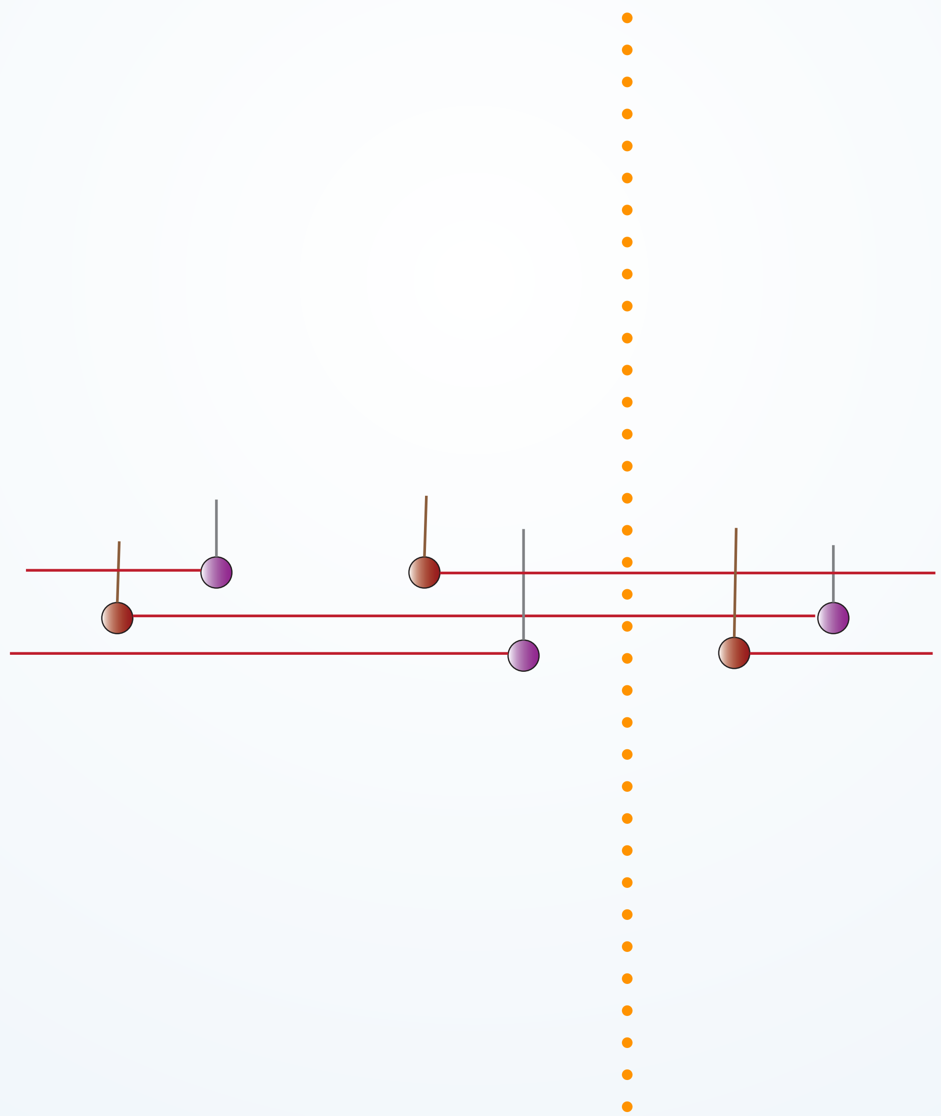
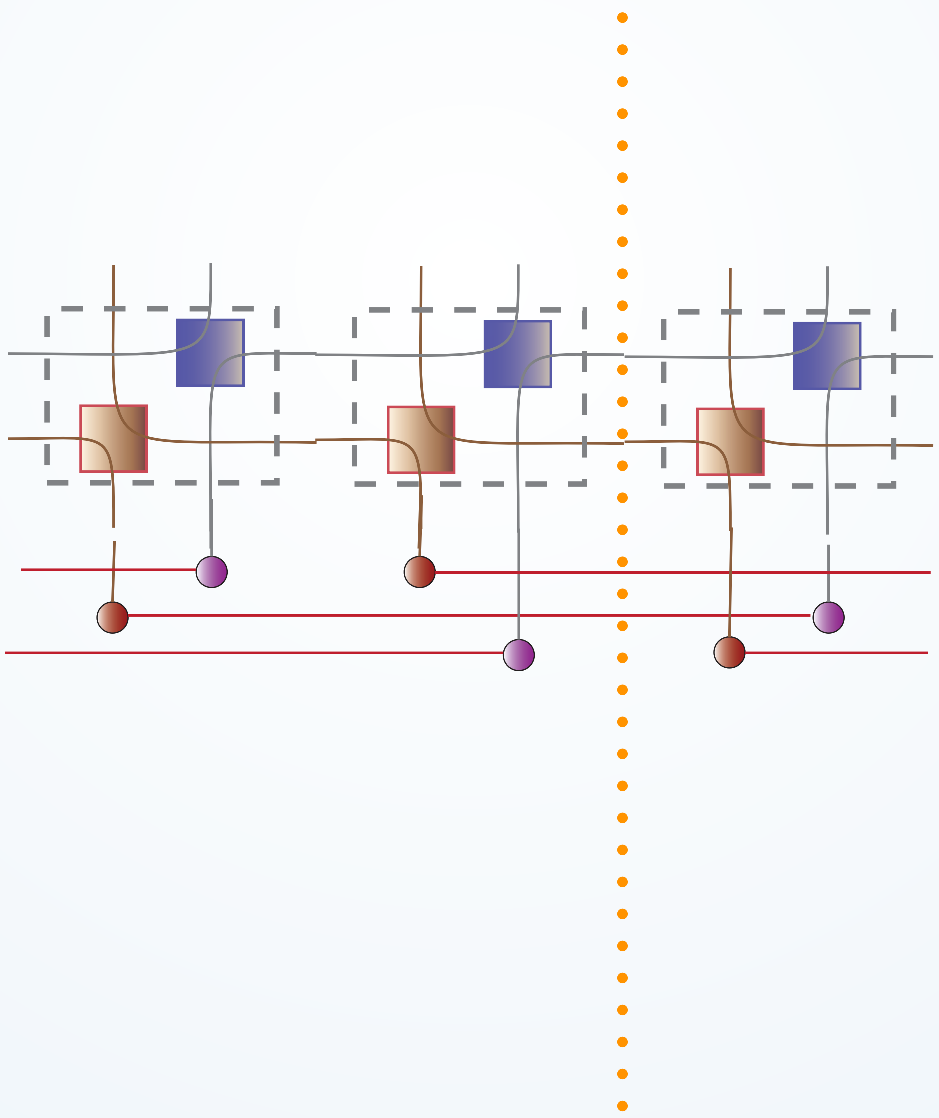


MCB, Hastings, Verstraete, Cirac, PRL 2009
Müller-Hermes, Cirac, MCB, NJP 2012

TOY MODEL TN



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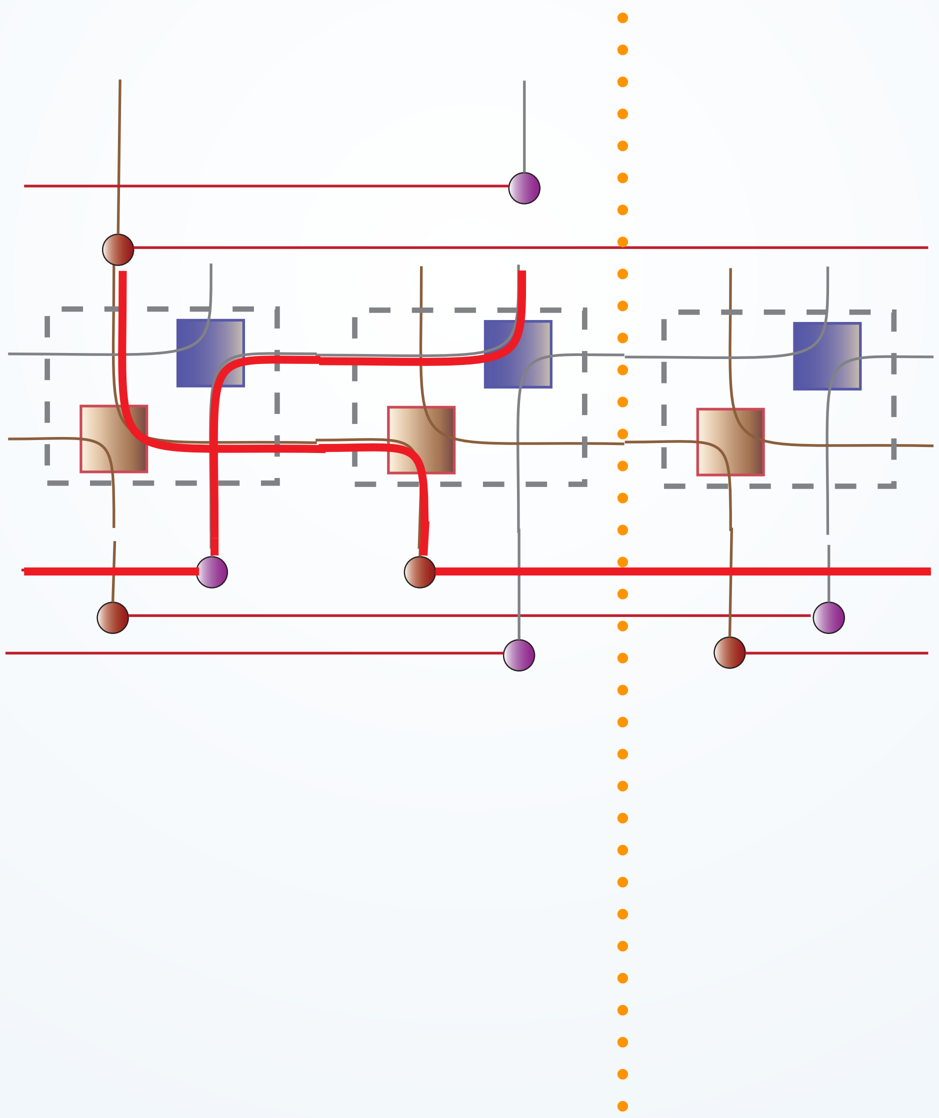
Müller-Hermes, Cirac, MCB, NJP 2012

TOY MODEL TN



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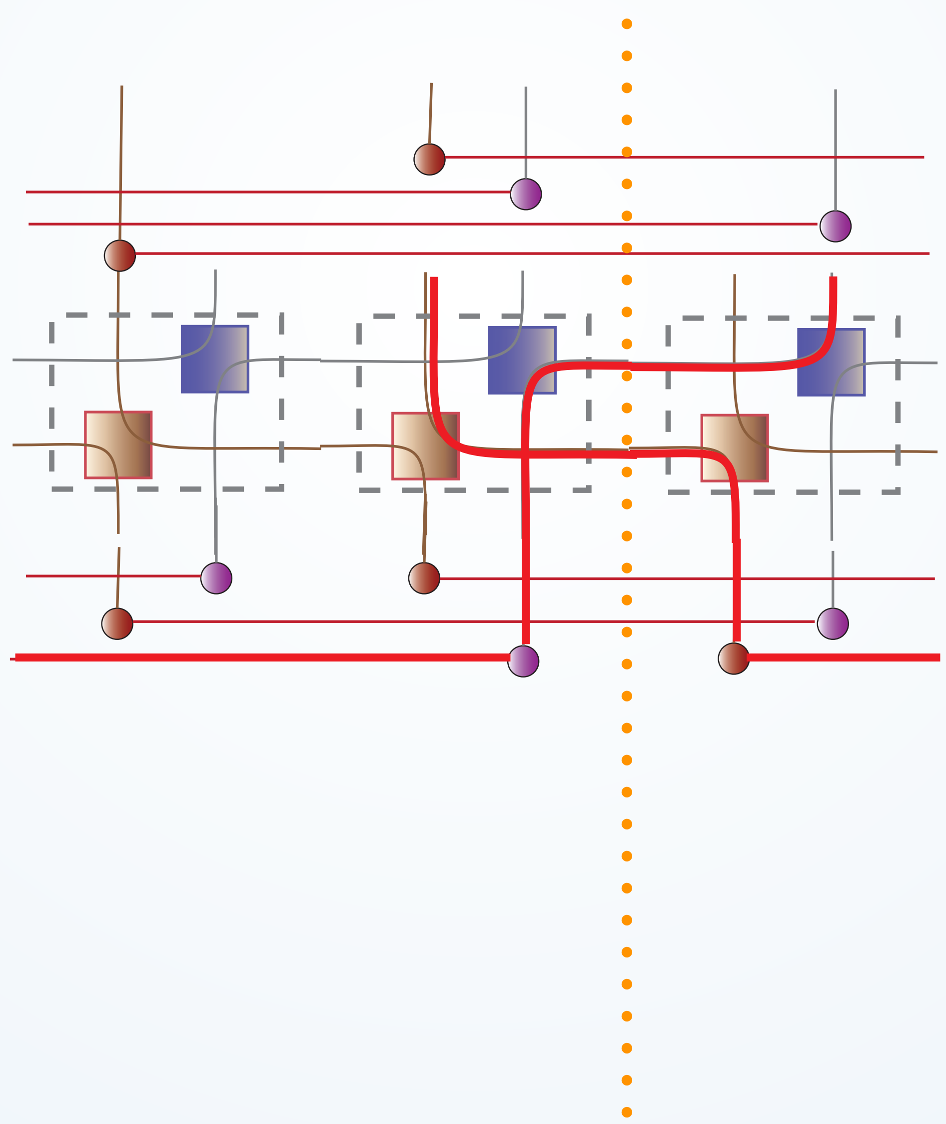
TOY MODEL TN



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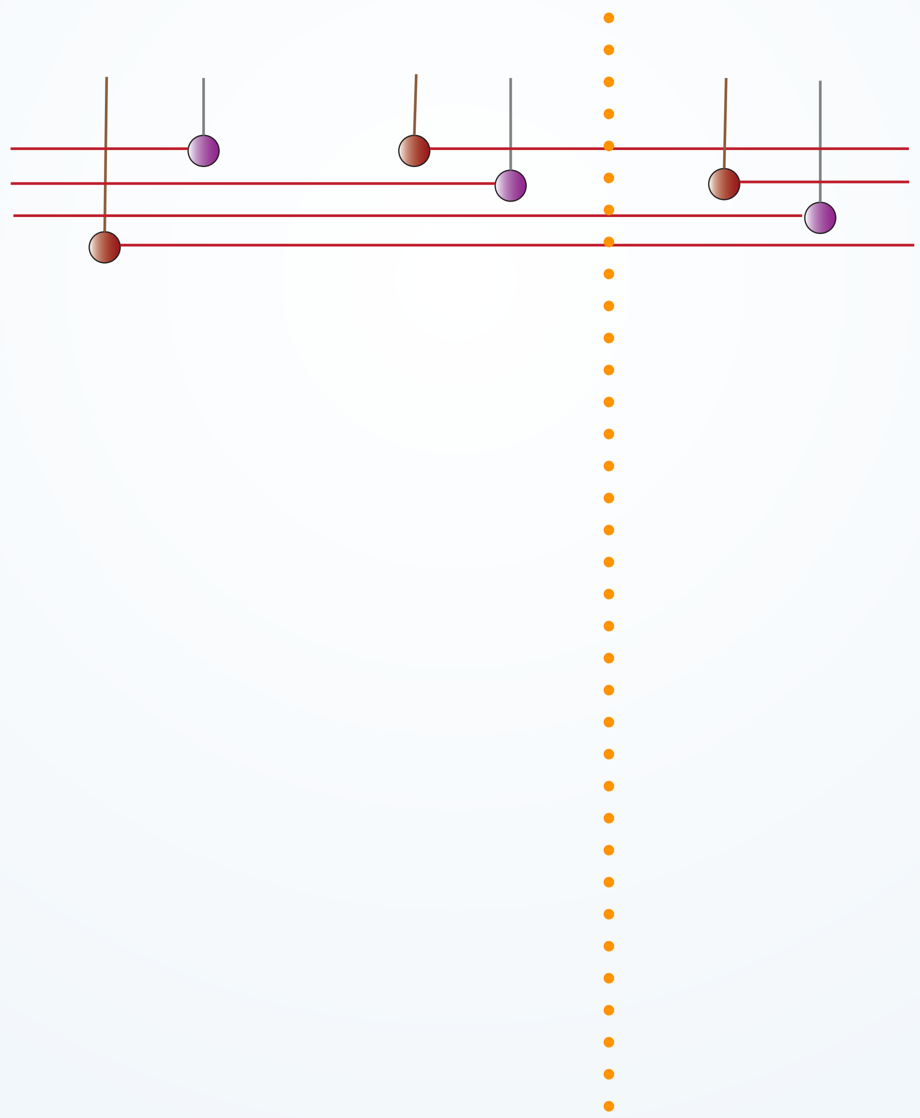
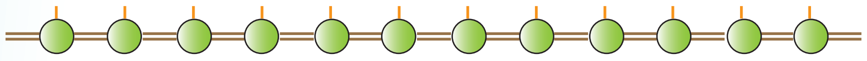


MCB, Hastings, Verstraete, Cirac, PRL 2009
Müller-Hermes, Cirac, MCB, NJP 2012

various evolution algorithms

evolving the (pure state) ansatz

Vidal, PRL 2003, PRL 2007
White, Feiguin, PRL 2004
Daley et al., 2004
Haegeman et al., 2011

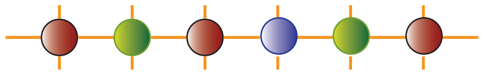


entanglement can grow fast!

Osborne, PRL 2006
Schuch et al., NJP 2008

evolving operators: Heisenberg picture

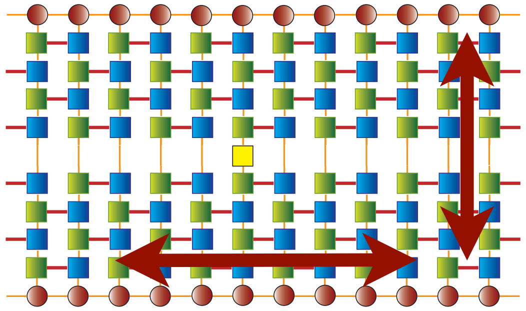
Hartmann et al, PRL 2009



also for mixed states
operator space entanglement

Prosen Pizorn, PRL 2008

observables as TN to contract



different *entanglement* quantities

MCB, Hastings, Verstraete, Cirac, PRL 2009
Müller-Hermes et al., NJP 2012
Hastings, Mahajan 2014
Frías-Pérez, MCB 2201.08402

THERMALIZATION

a question we would like to solve

THERMALIZATION

Closed quantum system initialized out of equilibrium

does it thermalize?

do local observables reach thermal equilibrium values?

independent of details of initial state?

predicted by statistical mechanics?

analytical techniques: exactly solvable systems

GGE

Rigol et al. PRL 2007
Cramer et al. PRL 2008
Calabrese et al. PRL 2011
Ilievski et al. PRL 2015

THERMALIZATION

Closed quantum system initialized out of equilibrium

does it thermalize?

do local observables reach thermal equilibrium values?

independent of details of initial state?

predicted by statistical mechanics?

non-integrable systems

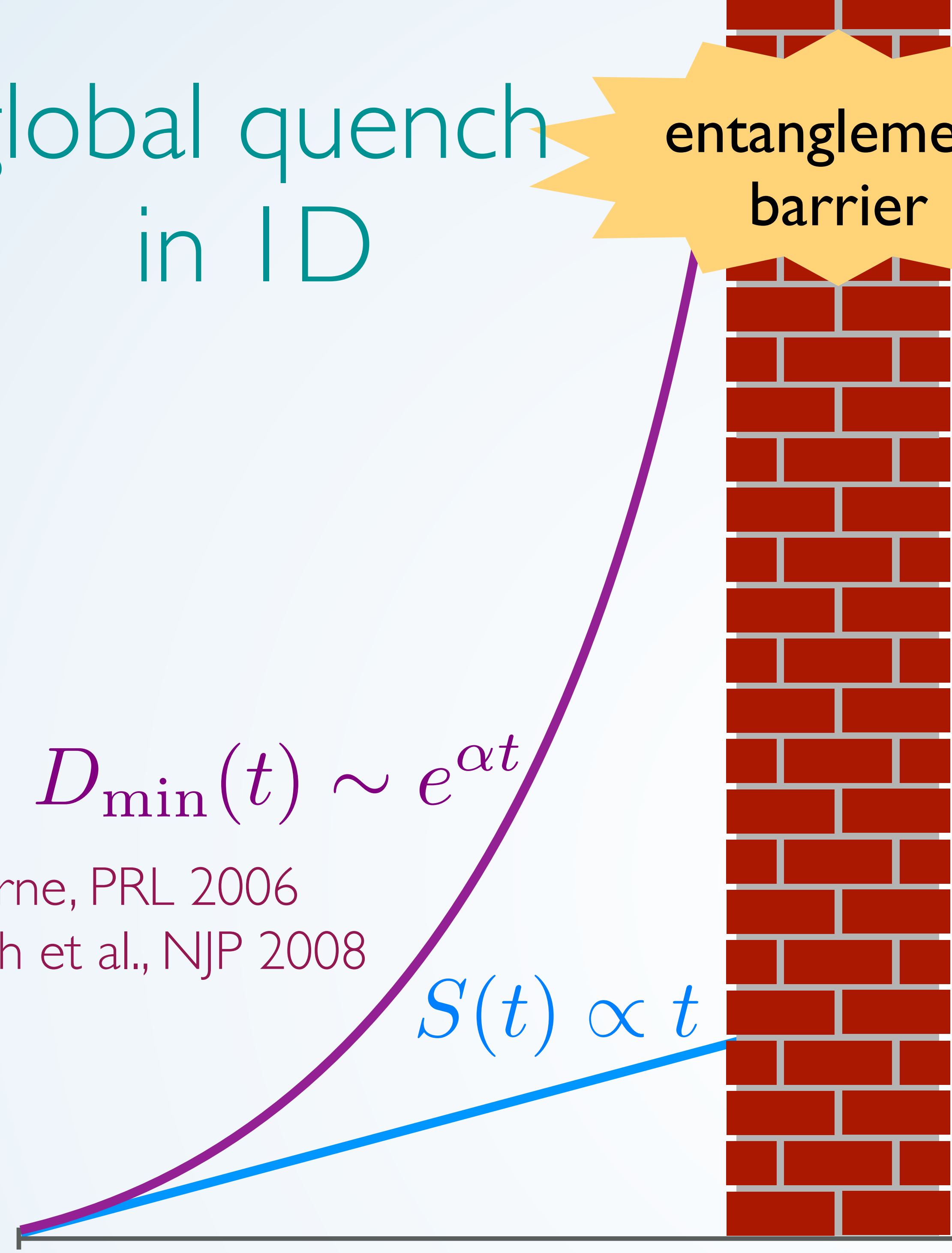
numerical simulations of real time evolution

Deutsch PRA91
Srednicki PRE94
Rigol et al. Nature 2008

ETH mechanism

time scales?

global quench in 1D



Osborne, PRL 2006
Schuch et al., NJP 2008

TNS challenge: getting around this limitation

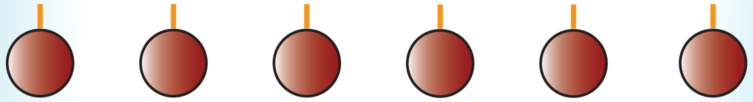
some recent progress
Dubai JPhysA 2017
Leviatan et al. 2017
White et al PRB 2018
Surace et al. 2018

work in progress!

Yang et al. PRL 2020
Lui et al PRX Quantum 2021
Frías-Pérez, MCB 2201.08402

$t = 0$

product state

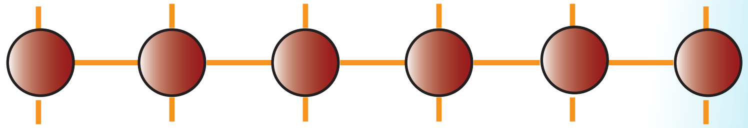


easy to write as MPS

local observables

$t = \infty$

thermal states

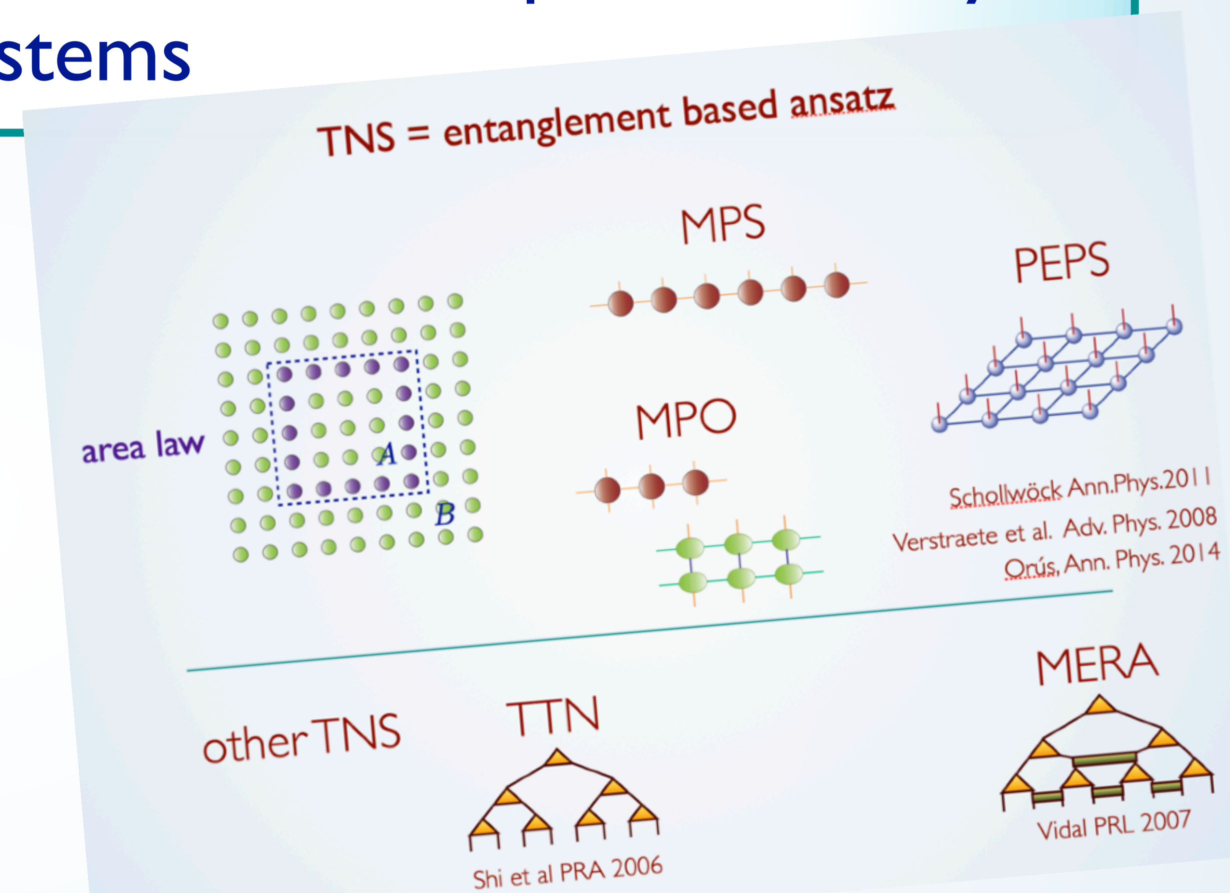


well approximated as MPO

to conclude…

TNS: (quantum inspired) methods to describe quantum many-body systems

NOTES:



efficient algorithms to find ground states, thermal states

simulating time evolution/arbitrary quantum circuits requires truncation ⇒ limited

T H A N K S !

Renormalization and tensor product states in spin chains and lattices, J. I. Cirac, F. Verstraete, J. Phys. A: Math. Theor. 42, 504004 (2009), arXiv:0910.1130

The density-matrix renormalization group in the age of matrix product states
Ulrich Schollwöck, Annals of Physics 326, 96 (2011), arXiv:1008.3477

Matrix Product States, Projected Entangled Pair States, and variational renormalization group methods for quantum spin systems, F. Verstraete, J.I. Cirac, V. Murg, Adv. Phys. 57,143 (2008), arXiv:0907.2796

A Practical Introduction to Tensor Networks: Matrix Product States and Projected Entangled Pair States, Roman Orús, Annals of Physics 349 (2014) 117, arXiv:1306.2164

Hand-waving and Interpretive Dance: An Introductory Course on Tensor Networks, Jacob C. Bridgeman, Christopher T. Chubb, J. Phys. A: Math. Theor. 50 223001 (2017), arXiv:1603.03039

The Tensor Networks Anthology: Simulation techniques for many-body quantum lattice systems, P. Silvi et al, SciPost Phys. Lect. Notes 8 (2019), arXiv:1710.03733

Introduction to Tensor Network Methods: Numerical simulations of low-dimensional many-body quantum systems, S. Montangero, Springer, 2018.

THANKS!