Spectral asymptotics for contracted tensor ensembles

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- Setting: d-th order N-dimensional real square symmetric tensors $T_{d,N} = \left(T_{d,N}(k_1, \dots, k_d)\right)_{(k_1,\dots,k_d) \in [N]^d} \in \mathcal{S}_{d,N} \subseteq \mathbb{R}^{N^d}$,

$$T_{d,N}(k_1, \dots, k_d) = T_{d,N}(k_{\sigma(1)}, \dots, k_{\sigma(d)})$$

- Basic question: how does the randomness of $T_{d,N}$ behave under repeated contractions?

- For $T_{d,N} \in \mathcal{S}_{d,N}$, $p \leq d$, and vectors $v_1, \dots, v_p \in \mathbb{R}^N$, we define the contracted tensor

$$T_{d,N}[v_1 \otimes \cdots \otimes v_p] \in \mathcal{S}_{d-p,N}$$

by

$$T_{d,N}[v_1 \otimes \cdots \otimes v_p](k_1, \dots, k_{d-p})$$

$$= \sum_{\ell_1, \dots, \ell_p} T_{d,N}(k_1, \dots, k_{d-p}, \ell_1, \dots, \ell_p)\, v_1(\ell_1) \cdots v_p(\ell_p)$$

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$$= \sum_{\ell_1, \dots, \ell_p} T_{d,N}(k_1, \dots, k_{d-p}, \ell_1, \dots, \ell_p)\, v_1(\ell_1) \cdots v_p(\ell_p)$$

Some observations:

- $T_{d,N} \in \mathcal{S}_{d,N}$ implies $T_{d,N}[v_1 \otimes \cdots \otimes v_p] \in \mathcal{S}_{d-p,N}$
- The order of the contracting vectors is immaterial
- The choice of contracted coordinates is also immaterial

- Basic question: how does the randomness of $T_{d,N}$ behave under repeated contractions?

- Example: $T_{1,N} \overset{d}{=} N(\hat{0}, Id_N)$, $v_N, w_N \in S^{N-1}$

$$\begin{pmatrix} T_{1,N}[v_N] \\ T_{1,N}[w_N] \end{pmatrix} \overset{d}{=} N(\hat{0}, \mathcal{K}), \qquad \mathcal{K} = \begin{pmatrix} \langle v_N, v_N \rangle & \langle v_N, w_N \rangle \\ \langle w_N, v_N \rangle & \langle w_N, w_N \rangle \end{pmatrix}$$

- Basic question: how does the randomness of $T_{d,N}$ behave under repeated contractions?

- Example: $T_{1,N} \overset{d}{=} (\underbrace{X_1, \ldots, X_N}_{\text{i.i.d.},\ \mu=0,\ \sigma^2=1})$, $\quad \underbrace{v_N, w_N}_{\|\cdot\|_\infty = o(1)} \in S^{N-1}$

$$\begin{pmatrix} T_{1,N}[v_N] \\ T_{1,N}[w_N] \end{pmatrix} \xrightarrow{d} N(\hat{0}, \mathcal{K}), \qquad \mathcal{K} = \lim_{N\to\infty} \begin{pmatrix} \langle v_N, v_N \rangle & \langle v_N, w_N \rangle \\ \langle w_N, v_N \rangle & \langle w_N, w_N \rangle \end{pmatrix}$$

- For our purposes, a Wigner matrix is a random real symmetric matrix $T_{2,N}$ such that:

  (1) the upper triangular entries are independent;

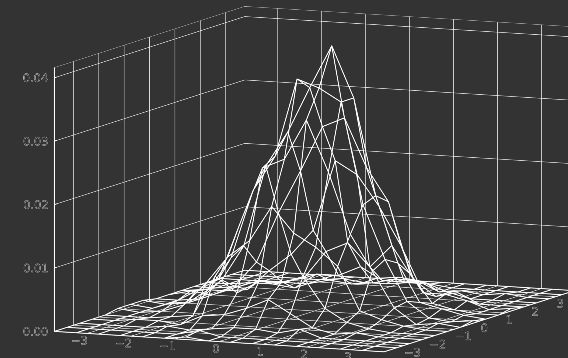
  (2) the off-diagonal entries are centered with variance $\frac{1}{2}$;

  (3) for any m,

$$\sup_{N \in \mathbb{N}} \sup_{k \le \ell} \mathbb{E}\left[|T_{2,N}(k,\ell)|^m \right] = C_m < \infty$$

- Example: $T_{2,N}$ $\overbrace{\text{Wigner}}^{\sigma^2 = 1 \text{ on diagonal}}$, $\quad \overbrace{v_N^{(1)}, v_N^{(2)}, w_N^{(1)}, w_N^{(2)}}^{\|\cdot\|_\infty = o(1)} \in S^{N-1}$

$$\begin{pmatrix} T_{2,N}\left[v_N^{(1)} \otimes v_N^{(2)}\right] \\ T_{2,N}\left[w_N^{(1)} \otimes w_N^{(2)}\right] \end{pmatrix} \xrightarrow{d} \mathcal{N}(\hat{0}, \mathcal{K}),$$



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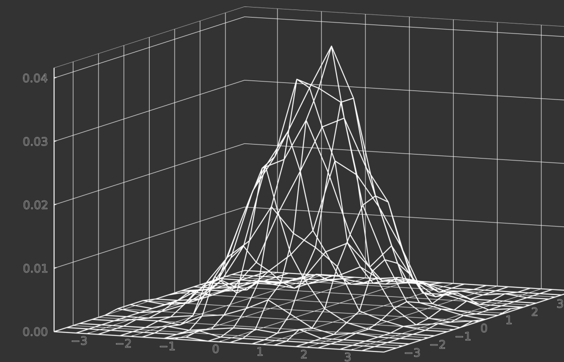
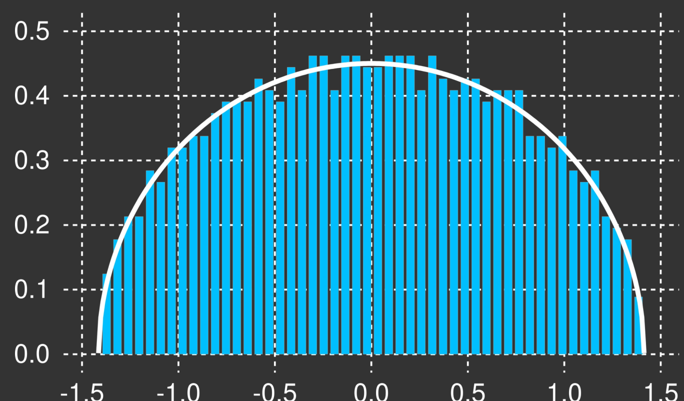


$$\mathcal{K} = \lim_{N \to \infty} \begin{pmatrix} \langle v_N^{(1)} \odot v_N^{(2)}, v_N^{(1)} \odot v_N^{(2)} \rangle & \langle v_N^{(1)} \odot v_N^{(2)}, w_N^{(1)} \odot w_N^{(2)} \rangle \\ \langle w_N^{(1)} \odot w_N^{(2)}, v_N^{(1)} \odot v_N^{(2)} \rangle & \langle w_N^{(1)} \odot w_N^{(2)}, w_N^{(1)} \odot w_N^{(2)} \rangle \end{pmatrix}$$

$$u_1 \odot \cdots \odot u_d = \frac{1}{d!} \sum_{\sigma \in S_d} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)} \in \mathcal{S}_{d,N}$$

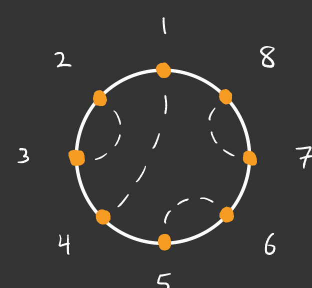
- (Wigner) The empirical spectral distribution of $W_N = \frac{1}{\sqrt{N}} T_{2,N}$ converges weakly almost surely to the semicircle distribution:

$$\mu(W_N) = \frac{1}{N} \sum_{k \in [N]} \delta_{\lambda_k(W_N)} \longrightarrow \frac{1}{\pi} \left(2 - x^2\right)_+^{\frac{1}{2}} dx$$



- (Voiculescu) Independent Wigner matrices $(W_N^{(i)})_{i\in I}$ are asymptotically free, converge in distribution to a multivariate semicircle $(s_i)_{i\in I} \overset{d}{=} SC(\hat{0}, \frac{1}{2}\mathrm{Id}_{\#(I)})$:

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\mathrm{Tr}\left[W_N^{(i_1)}\cdots W_N^{(i_m)}\right]\right] = \sum_{\pi\in \underbrace{NC_2(m)}_{\text{non-crossing}}} \prod_{\{j,k\}\in\pi} \underbrace{\frac{1}{2}\mathrm{Id}_{\#(I)}(i_j, i_k)}_{\mathcal{K}(i_j,\, i_k)}$$



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cf. $(X_i)_{i\in I} \overset{d}{=} N(\hat{O}, \mathcal{K})$:

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- Back to tensors: if $T_{d,N}$ is a random symmetric tensor, the corresponding contracted tensor ensemble (GCC21) is the family of random matrices $\{T_{d,N}[u^{\otimes d-2}]\}_{u \in S^{N-1}}$
- What kind of randomness? A canonical distribution:

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- (GCC21) For any sequence of unit vectors $u_N \in S^{N-1}$, the empirical spectral distribution of $W_N = \frac{1}{\sqrt{N}} T_{3,N}[u_N]$ converges weakly almost surely to the semicircle distribution with $\sigma^2 = \frac{1}{6}$

GOTE

- In general, W_N is not a Wigner matrix:

$$T_{3,N}[u_N](j,k) = \sum_{\ell} T_{3,N}(j,k,\ell)\, u_N(\ell)$$

- Proof relies on Stein's method and $d = 3$

- Question 1: what about higher order $d \geq 4$?

$$T_{3,N}[u_N](j,k) = \sum_{\ell} T_{3,N}(j,k,\ell)\, u_N(\ell)$$

$$T_{4,N}[u_N^{\otimes 2}](j,k) = \sum_{\ell_1,\ell_2} T_{4,N}(j,k,\ell_1,\ell_2)\, u_N(\ell_1)\, u_N(\ell_2)$$

- Question 2: universality for general tensor distributions?

- Question 3: general contractions $u_N^{(1)} \otimes \cdots \otimes u_N^{(d-2)} \neq u_N^{\otimes d-2}$?

- Question 4: joint behavior of $\left\{ T_{d,N}[u_N^{\otimes d-2}] \right\}_{u_N \in S^{N-1}}$?

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(2') entries with $\#(k_1, \dots, k_d) \geq 3$ have variance $\binom{d}{b_1, \dots, b_N}^{-1}$;

(3) for any m,

$$\sup_{N \in \mathbb{N}} \sup_{\vec{k}} \mathbb{E}\left[\left| T_{d,N}(k_1, \dots, k_d) \right|^m \right] = C_m < \infty$$

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  (2′) entries with $\#(k_1, \dots, k_d) \geq 3$ have **variance** $\boxed{\frac{1}{Z_{d,N}} e^{-\|H\|_F^2/2}\, dH} \to \binom{d}{b_1, \dots, b_N}^{-1}$;

  (3) for any m,

$$\sup_{N \in \mathbb{N}} \sup_{\vec{k}} \mathbb{E}\left[|T_{d,N}(k_1, \dots, k_d)|^m \right] = C_m < \infty$$

- For vectors $u_1, \dots, u_d \in \mathbb{R}^N$, define the symmetrization

$$u_1 \odot \cdots \odot u_d = \frac{1}{d!} \sum_{\sigma \in S_d} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)} \in \mathcal{S}_{d-p, N}.$$

- For any sequence of families of unit vectors $\{u_N^{(i,j)}\}_{i \in I, j \in [d-2]}$, let $\mathcal{K}^{(N)} = (\mathcal{K}^{(N)}(i, i'))_{i, i' \in I}$ be the rescaled Gram matrix of the symmetrizations:

$$\mathcal{K}^{(N)}(i, i') = \frac{1}{d(d-1)} \left\langle u_N^{(i,1)} \odot \cdots \odot u_N^{(i,d)}, \ u_N^{(i',1)} \odot \cdots \odot u_N^{(i',d)} \right\rangle.$$

- (AGV21) Let $T_{d,N}$ be a Wigner tensor and define $(W_N^{(i)})_{i\in I} = \left(\frac{1}{\sqrt{N}} T_{d,N}[u_N^{(i,1)} \otimes \cdots \otimes u_N^{(i,d-2)}]\right)_{i\in I}$. Then $(W_N^{(i)})_{i\in I}$ converges in distribution a.s. iff the limits

$$\mathcal{K}(i,i') = \lim_{N\to\infty} \mathcal{K}^{(N)}(i,i')$$

exist, in which case $(W_N^{(i)})_{i\in I} \to SC(\hat{O}, \mathcal{K})$.

- $\left(W_N^{(i)}\right)_{i\in I} = \left(\frac{1}{\sqrt{N}} T_{d,N}\left[u_N^{(i,1)} \otimes \cdots \otimes u_N^{(i,d-2)}\right]\right)_{i\in I}$

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- $\left(W_N^{(i)}\right)_{i\in I} \to SC(\hat{O}, \mathcal{K})$ in distribution a.s.

- What about a single matrix $W_N^{(i)}$?

- First, assume $u_N^{(i,1)} \otimes \cdots \otimes u_N^{(i,d-2)} = u_N^{\otimes d-2}$. Then

$$\mathcal{K}(i,i) = \frac{1}{d(d-1)}$$

- $\left(W_N^{(i)}\right)_{i\in I} = \left(\frac{1}{\sqrt{N}} T_{d,N}\left[u_N^{(i,1)} \otimes \cdots \otimes u_N^{(i,d-2)}\right]\right)_{i\in I}$

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- $\left(W_N^{(i)}\right)_{i\in I} \to SC(\hat{O}, \mathcal{K})$ in distribution a.s.

- What about a single matrix $W_N^{(i)}$?

- Recall the permanent identity

$$\left\langle u_N^{(i,1)} \odot \cdots \odot u_N^{(i,d)} , u_N^{(i',1)} \odot \cdots \odot u_N^{(i',d)} \right\rangle = \frac{1}{(d-2)!} \operatorname{per}\left[\left(\left\langle u_N^{(i,j)}, u_N^{(i',k)} \right\rangle\right)_{j,k\in[d-2]}\right]$$

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- $\left(W_N^{(i)}\right)_{i\in I} \approx SC(\hat{O}, \mathcal{K}_N)$: for any finite subset $I_0 \subseteq I$, exponent rate M, moment threshold m_0, and error $\varepsilon > 0$, there is a constant $C = C(d, \#(I_0), M, m_0, \varepsilon)$ such that

$$\mathbb{P}\left[\max_{\substack{m \le m_0 \\ i_1,\dots,i_m \in I_0}} \left| \frac{1}{N} \mathrm{Tr}\left[W_N^{(i_1)} \cdots W_N^{(i_m)}\right] - \sum_{\pi \in NC_2(m)} \prod_{\{j,k\}\in\pi} \mathcal{K}_N(i_j, i_k) \right| > \varepsilon\right] < \frac{C}{N^M}$$

- $$\left(W_N^{(i)}\right)_{i\in I} = \left(\frac{1}{\sqrt{N}} T_{d,N}\left[u_N^{(i,1)} \otimes \cdots \otimes u_N^{(i,d-2)}\right]\right)_{i\in I}$$

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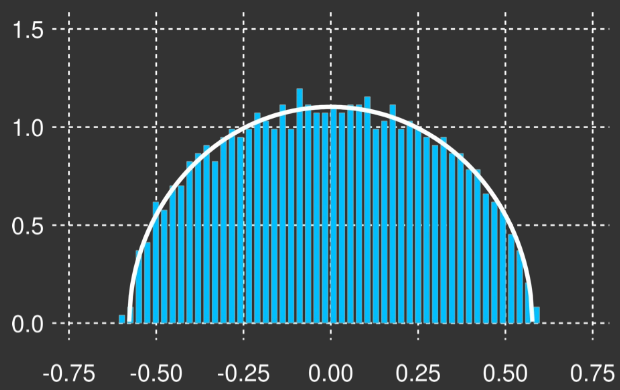

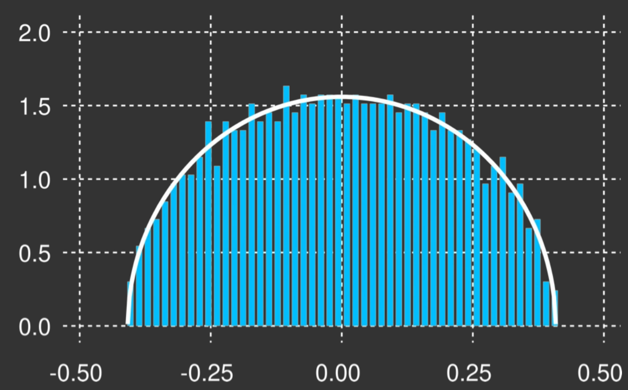

- $\left(W_N^{(i)}\right)_{i \in I} = \left(\frac{1}{\sqrt{N}} T_{d,N}\left[u_N^{(i,1)} \otimes \cdots \otimes u_N^{(i,d-2)} \right] \right)_{i \in I}$

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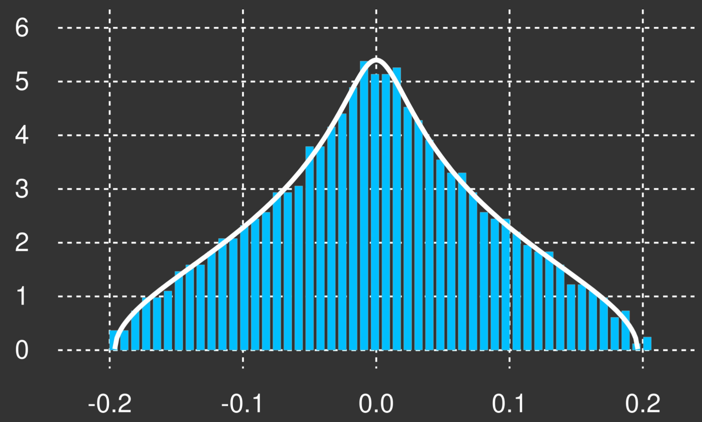

- Is this covered by some result for dependent random matrices? Thankfully, no.
- (SSB05) Partitioned entries (but constrained block size)
- (GNT15) Conditional centeredness ($\mathbb{E}[X_{i,j} \mid \mathcal{F}_{i,j}] = 0$)
- (BMP15) Array representation ($X_{i,j} = g(Y_{i-k,\ell-j} : (k,\ell) \in \mathbb{Z}^2)$)
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Proof techniques:

Moment method for tensor contractions

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- (Male 2011)

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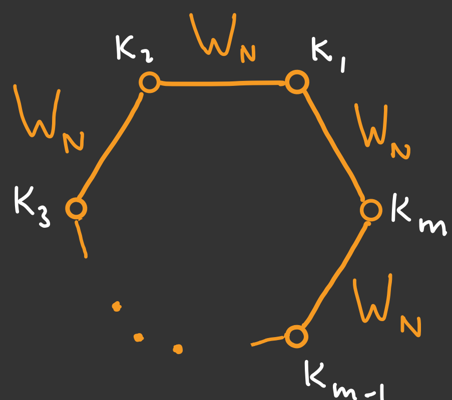

$$\longleftrightarrow \quad W_N(k_1,k_2)\cdots W_N(k_m,k_1)$$

Warmup: $d=2$

- Usual Wigner matrix case $W_N = \frac{1}{\sqrt{N}} T_{2,N} =: \frac{1}{\sqrt{N}} T_N$.

- Understand

$$\frac{1}{N}\mathbb{E}\left[\operatorname{Tr} W_N^m\right] = \frac{1}{N}\sum_{k_1,\dots,k_m\in[N]} \mathbb{E}\left[W_N(k_1,k_2)\cdots W_N(k_m,k_1)\right]$$

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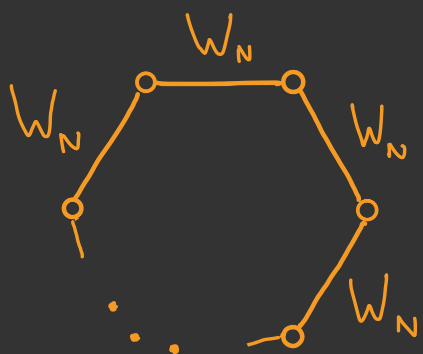
$G =$ 

- $$t(G) := \frac{1}{N} \sum_{\phi: V \to [N]} \mathbb{E}\left[W_N(\phi(v_1), \phi(v_2)) \cdots W_N(\phi(v_m), \phi(v_1)) \right]$$

- $$\tau(G) := \frac{1}{N} \sum_{\phi: V \to [N]} \mathbb{E}\left[W_N(\phi(v_1), \phi(v_2)) \cdots W_N(\phi(v_m), \phi(v_1)) \right]$$

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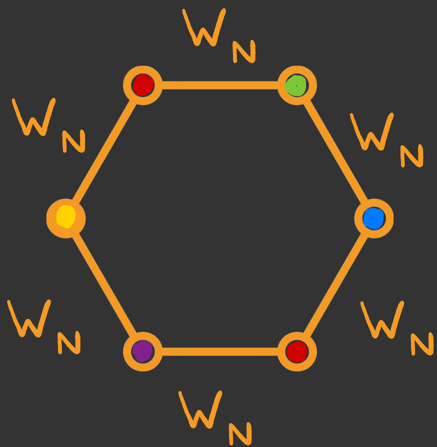
- $G =$ 

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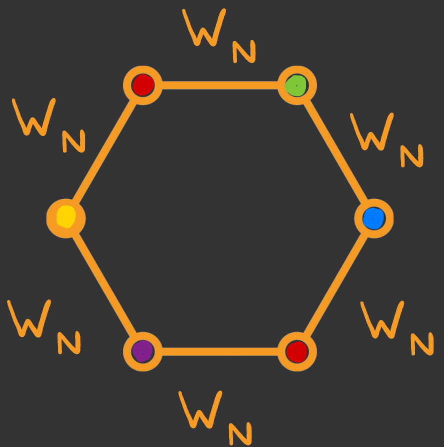
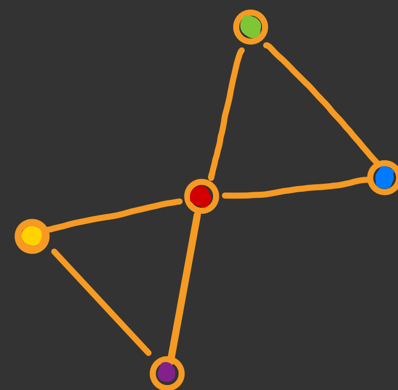
- $G =$



$\longmapsto$ $G^{\pi} =$



For $\pi \in P(V)$

- $$\tau(G) := \frac{1}{N} \sum_{\phi : V \to [N]} \mathbb{E}\left[W_N(\phi(v_1), \phi(v_2)) \cdots W_N(\phi(v_m), \phi(v_1)) \right]$$

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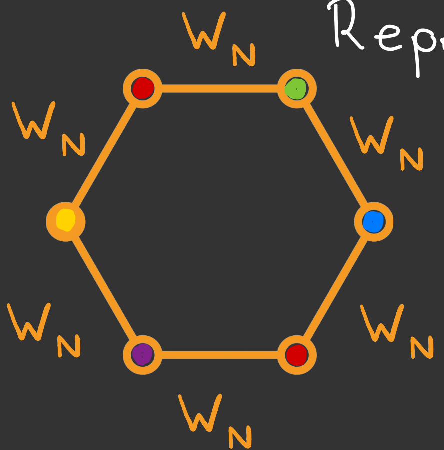
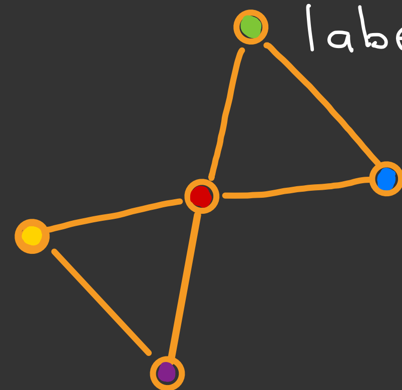
- $G =$ 

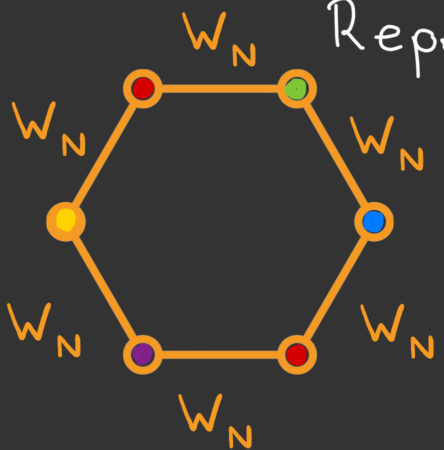
Repeated labels $\longmapsto$ $G^\pi =$ 

Different labels

For $\pi \in P(V)$

• $$\tau(G) := \frac{1}{N} \sum_{\phi : V \to [N]} \mathbb{E}\left[W_N(\phi(v_1), \phi(v_2)) \cdots W_N(\phi(v_m), \phi(v_1)) \right]$$

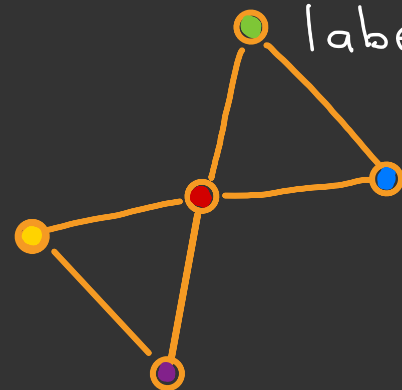
$$= \frac{1}{N} \mathbb{E}\left[\mathrm{Tr}\, W_N^m \right]$$

• $G =$ 

Repeated labels

$\longmapsto$

For $\pi \in P(V)$

$G^\pi =$ 

Different labels

• Then we can decompose $\tau(G) = \sum_{\pi \in P(V)} \tau^0(G^\pi)$

where $$\tau^0(G) = \frac{1}{N} \sum_{\phi : V^\pi \hookrightarrow [N]} \mathbb{E}\left[W_N(\phi(v_1), \phi(v_2)) \cdots W_N(\phi(v_m), \phi(v_1)) \right]$$

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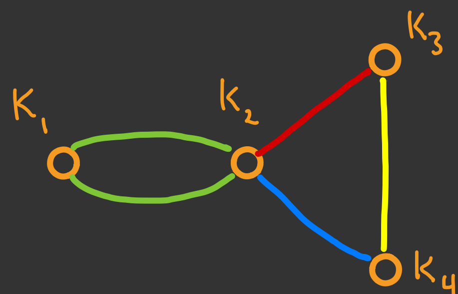

$$\mathbb{E}\left[W_N^2(k_1,k_2)\, W_N(k_2,k_3)\, W_N(k_3,k_4)\, W_N(k_4,k_2)\right]$$

$$= \mathbb{E}\left[W_N^2(k_1,k_2)\right] \mathbb{E}\left[W_N(k_2,k_3)\right] \mathbb{E}\left[W_N(k_3,k_4)\right] \mathbb{E}\left[W_N(k_4,k_2)\right]$$

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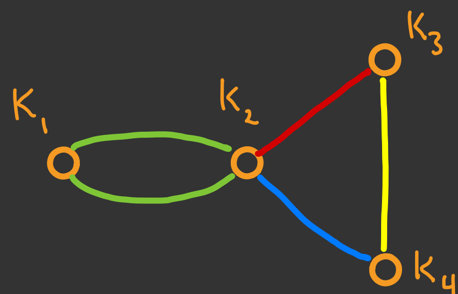

$$\mathbb{E}[W_N^2(k_1,k_2)\, W_N(k_2,k_3)\, W_N(k_3,k_4)\, W_N(k_4,k_2)]$$

$$= \mathbb{E}[W_N^2(k_1,k_2)]\ \cancelto{0}{\mathbb{E}[W_N(k_2,k_3)]}\ \cancelto{0}{\mathbb{E}[W_N(k_3,k_4)]}\ \cancelto{0}{\mathbb{E}[W_N(k_4,k_2)]}$$

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$$= \frac{1}{N^{\frac{m}{2}+1}} \sum_{\phi : V^{\pi} \hookrightarrow [N]} O(1)$$

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$$\left.\begin{array}{l} (A) \Longrightarrow |\pi| \leq m/2 + 1 \\[2ex] (B) \Longrightarrow |\pi| \geq m/2 + 1 \end{array}\right\} \xRightarrow{\text{Combinatorics}} \begin{array}{l} G^\pi \text{ is a double} \\ \text{tree, in which} \\ \text{case } \tau^\circ(G^\pi) = \sigma^m \end{array}$$

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G^π is a double tree, in which case $\tau^\circ(G^\pi) = \sigma^m$

- Hence $\tau(G) \sim \sum_{\substack{\pi \in P(V) \\ G^\pi \text{ is a double tree}}} \tau^\circ(G^\pi)$

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- Hence $\tau(G) \sim \sum_{\substack{\pi \in P(V) \\ G^\pi \text{ is a double tree}}} \tau^\circ(G^\pi) \overset{\text{Combinatorics}}{=} \begin{cases} 0 & \text{if } m \text{ is odd} \\ \sigma^{2K} C_K & \text{if } m = 2K \end{cases}$

Graphs of tensors: $d > 2$

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- One option is to work directly with the matrices

$$\left(\frac{1}{\sqrt{N}} T_{d,N} \left[u_N^{(i,1)} \otimes \cdots \otimes u_N^{(i,d-2)} \right] \right)_{i \in I}$$

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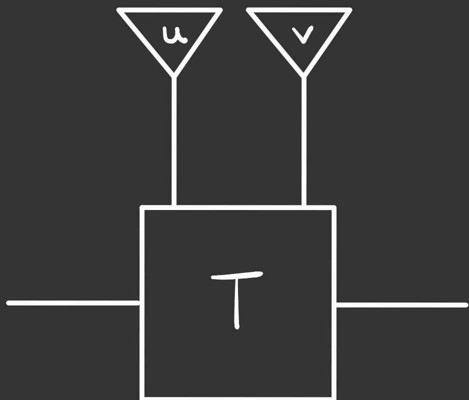
The graph analysis exploits independence of the entries

Graphs of tensors: $d > 2$

- (Penrose) Tensors are represented as shapes such as boxes and triangles; indices are denoted by lines.



$T[u \otimes v]$

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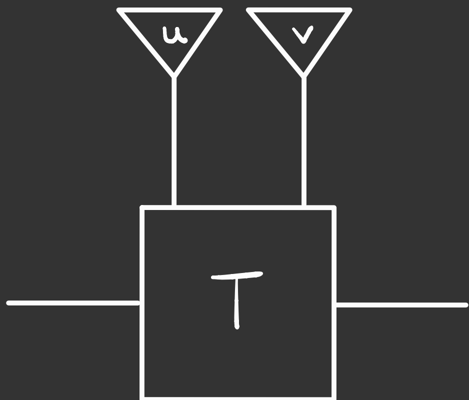

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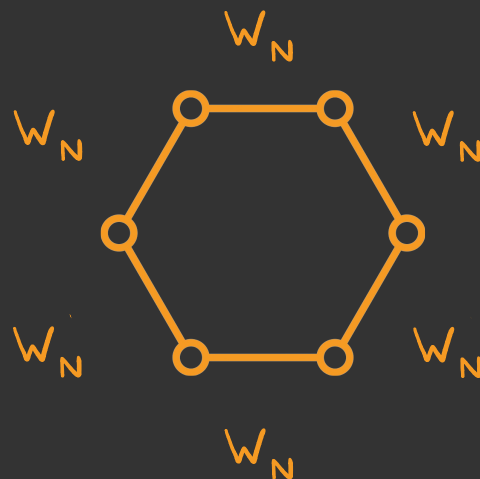
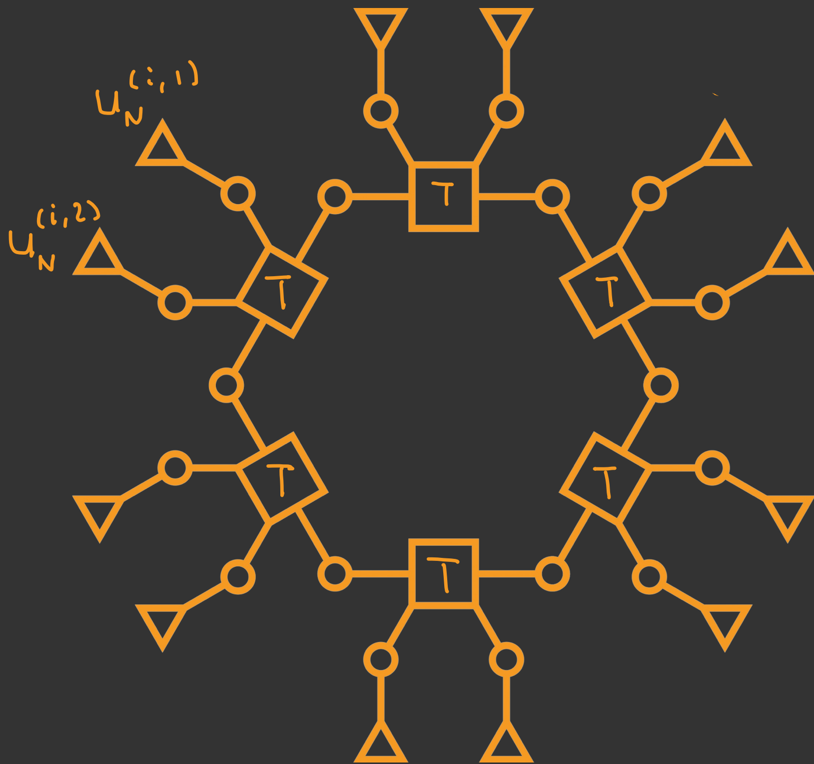


- (RMT) Matrices represented by lines; indices by circles.

Graphs of tensors: $d > 2$

$G :=$



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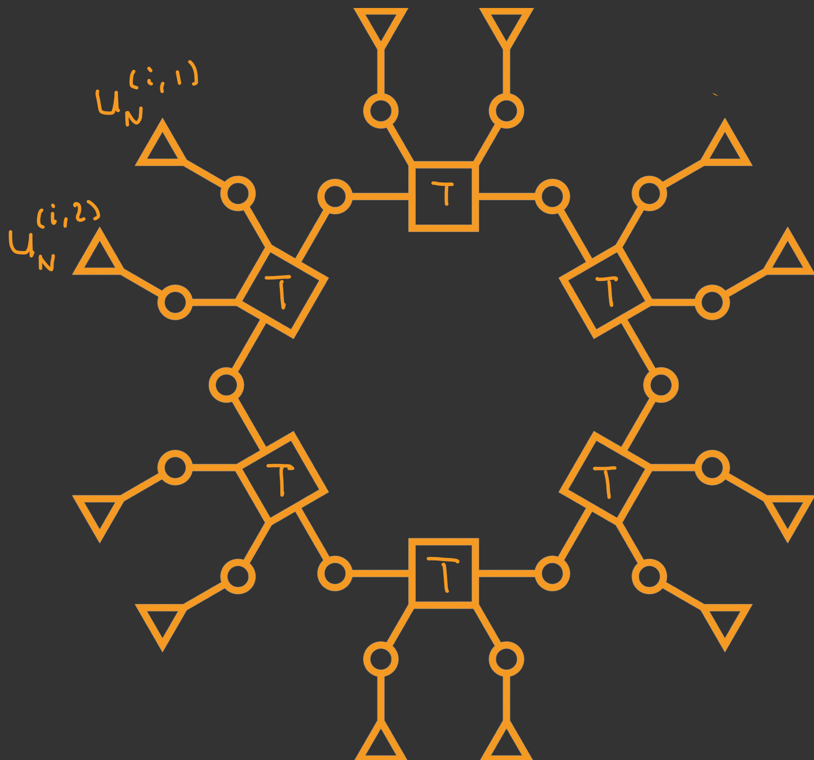
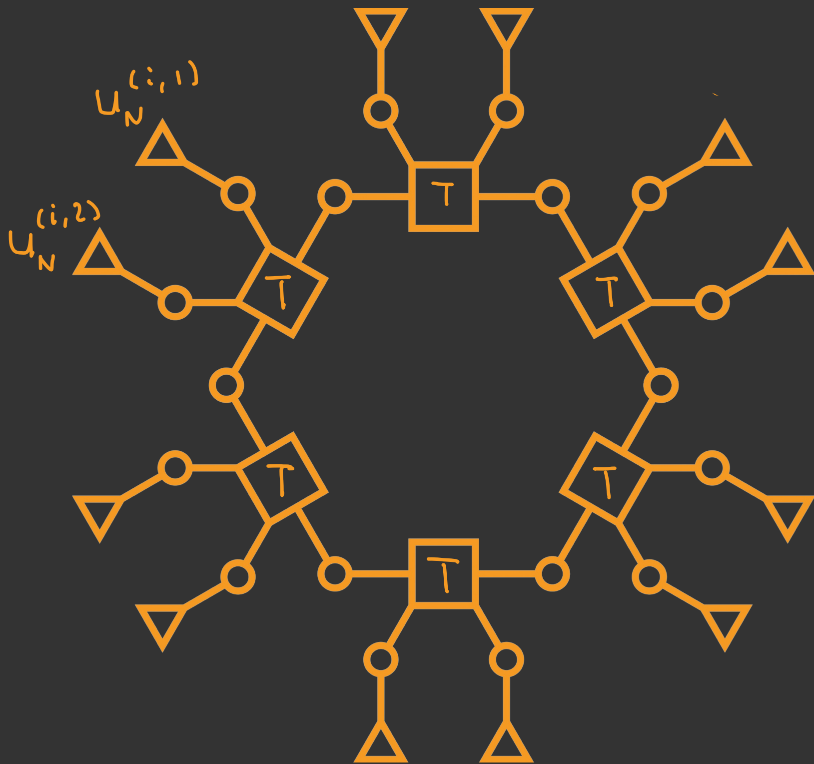
$G :=$ 

- $V(G) = V \sqcup W$

  V for indices (circles)

  W for tensors (boxes or triangles)

Graphs of tensors: $d > 2$

$G :=$ 

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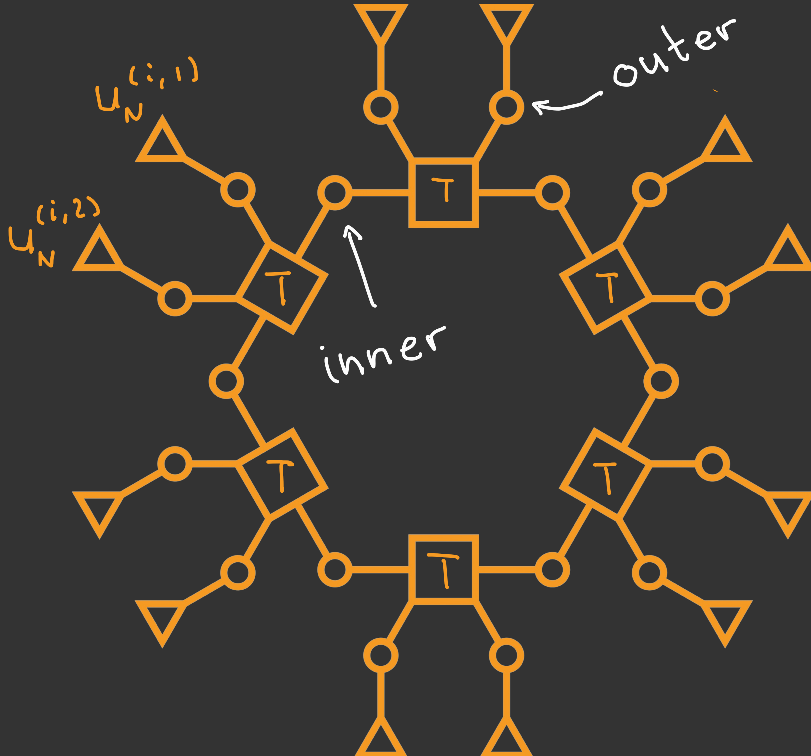
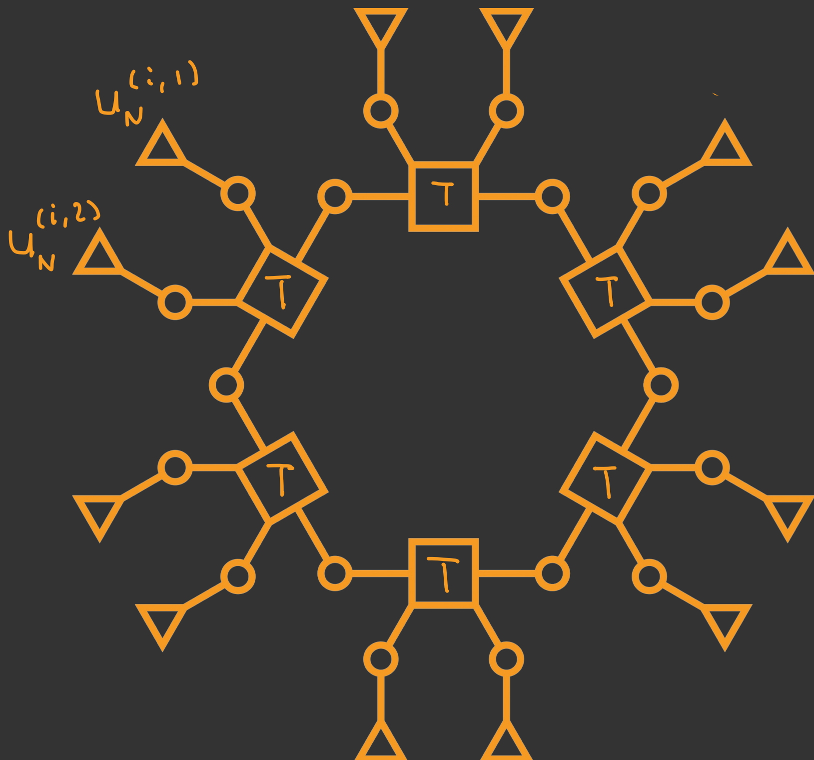
$G :=$ 

- $V(G) = V \sqcup W$
- $V = V_{in} \sqcup V_{out}$

Graphs of tensors: $d > 2$

$G :=$ 

- $V(G) = V \sqcup W$
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- $W = W_1 \sqcup W_2$

W_1 boxes

W_2 triangles

Graphs of tensors: $d > 2$

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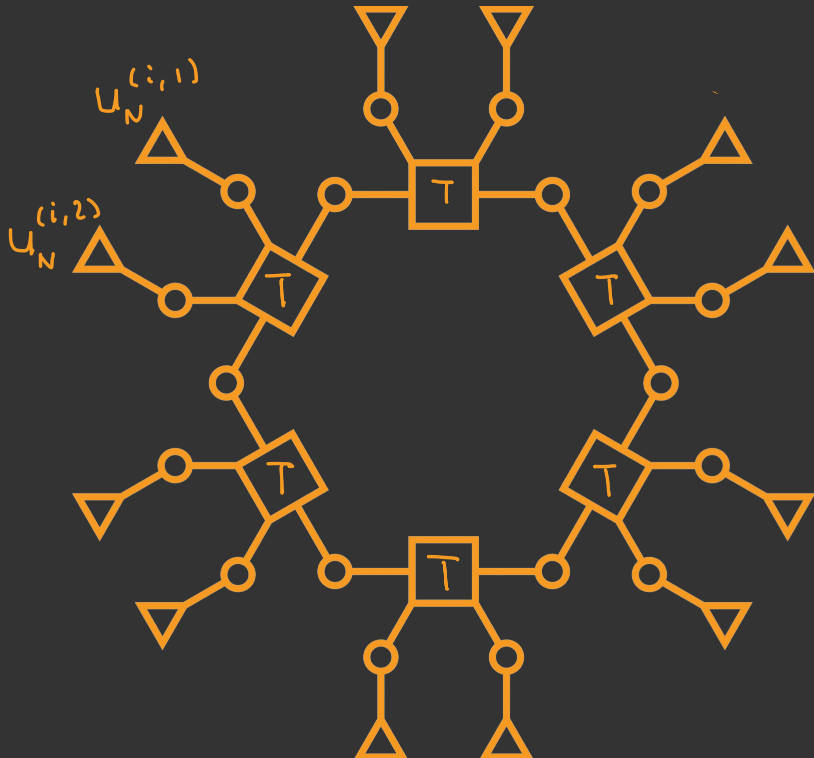

- $V(G) = V \sqcup W$
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W_1 boxes

W_2 triangles

$$\tau(G) = \frac{1}{N^{m/2+1}} \sum_{\phi : V \to [N]} \mathbb{E}\Big[\prod_{w_1 \in W_1} T_{d,N}(w_1 \mid \phi) \Big] \prod_{w_2 \in W_2} u_N(w_2 \mid \phi)$$

Graphs of tensors: $d>2$

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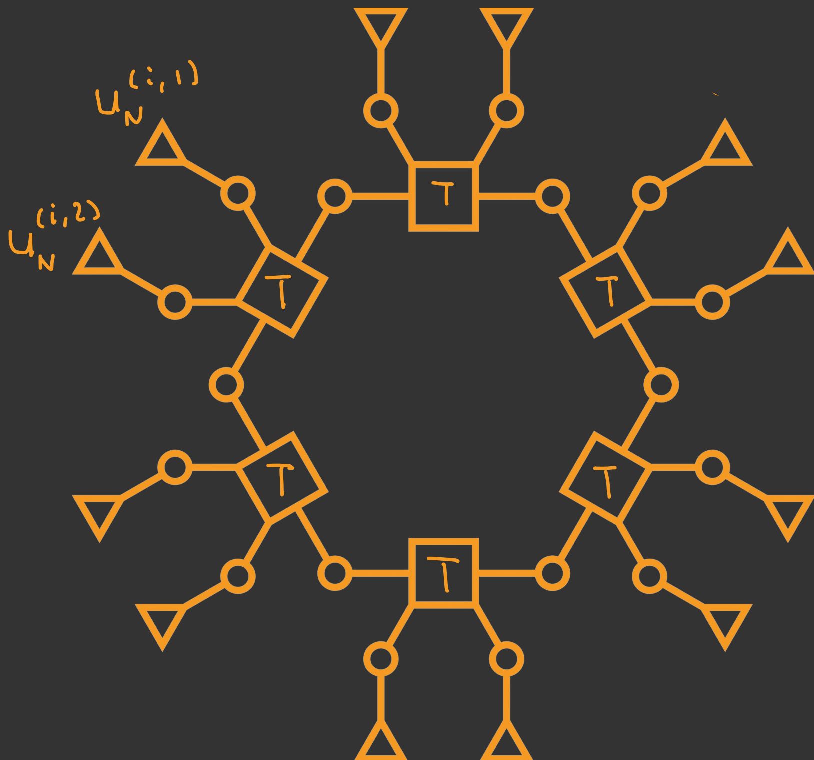

- $V(G) = V \sqcup W$
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$$\tau^\circ(G^\pi) = \frac{1}{N^{m/2+1}} \sum_{\phi : V^\pi \hookrightarrow [N]} \mathbb{E}\Big[\prod_{w_1 \in W_1} T_{d,N}(w_1 | \phi) \Big] \prod_{w_2 \in W_2} u_N(w_2 | \phi)$$

$\forall \pi \in P(V)$

Complications

- Again we have $\tau(G) = \sum_{\pi \in P(V)} \tau^{\circ}(G^{\pi})$ but now analyzing each $\lim_{N \to \infty} \tau^{\circ}(G^{\pi})$ is harder.

Complications

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- Complication A: Independence + centeredness is not as effective when ruling out partitions

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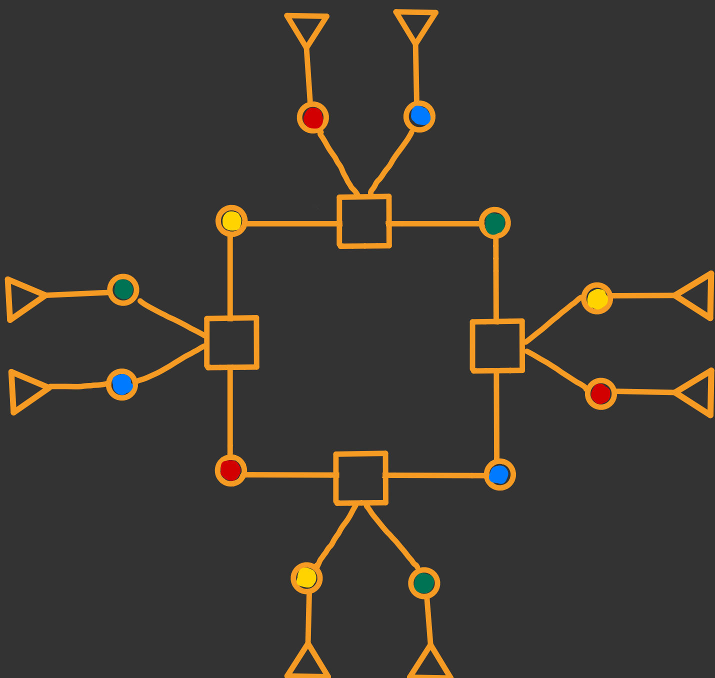
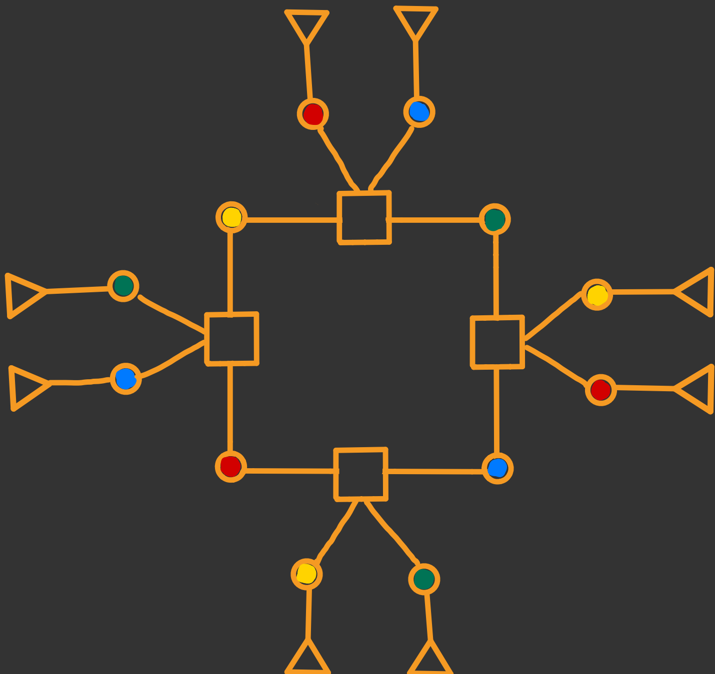


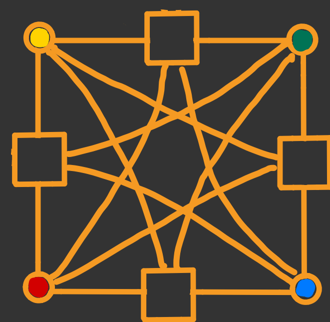
$$\mathbb{E}[T_{d,N}(k_1, k_2, k_3, k_4)^4] \neq 0$$

Complications

- Complication (A): Independence + centeredness is not as effective when ruling out partitions



$$\longmapsto$$



$$\mathcal{T}^0(G^\pi) \neq 0 \quad (\text{for } d=4)$$

$$\mathbb{E}\left[T_{d,N}(k_1, k_2, k_3, k_4)^4 \right] \neq 0$$

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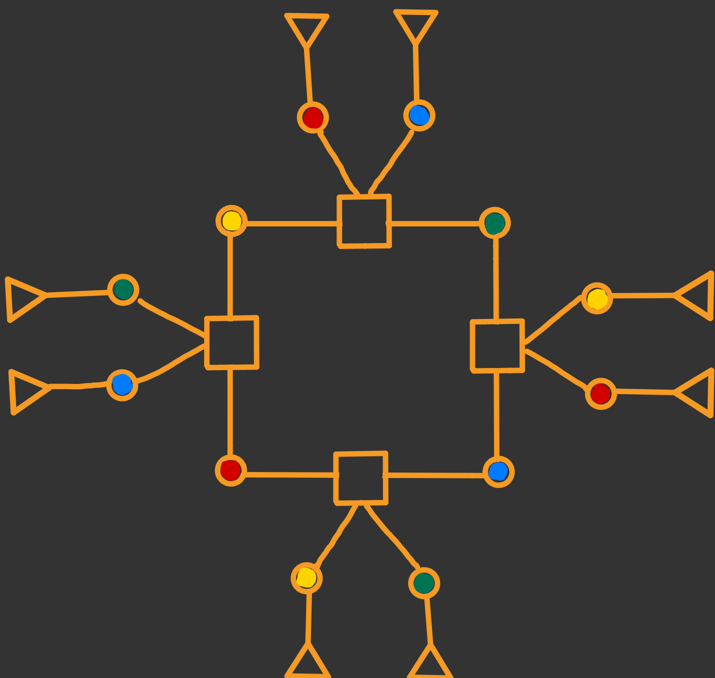

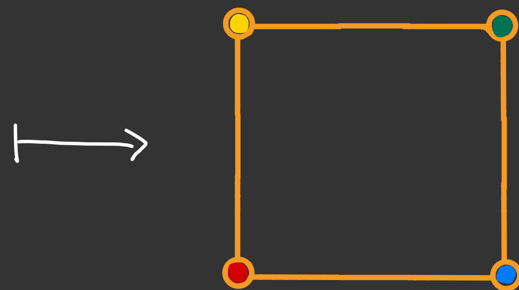

$\tau^{0}(G^{\pi}) \neq 0$ (for $d=4$)

$\tau^{0}(G^{\pi}) = 0$ (for $d=2$)

$\mathbb{E}[\, T_{d,N}(k_1, k_2, k_3, k_4)^4 \,] \neq 0$

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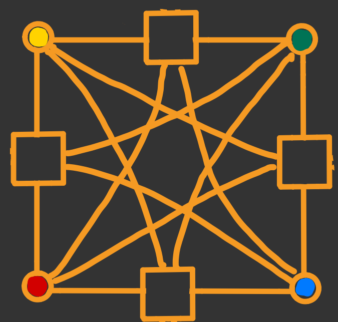
E.g. Take $m=4$, $d=4$ and π as follows

$G^\pi =$ 

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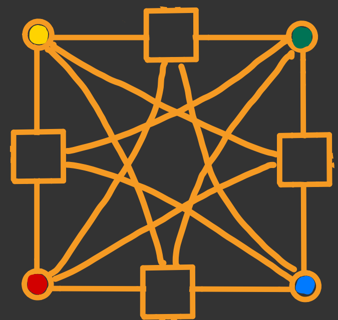
$G^\pi =$  , then using that $|u_N^{(i,j)}(k)| = O(1)$

$$\tau^0(G^\pi) = \frac{1}{N^{m/2+1}} \sum_{\phi : V^\pi \hookrightarrow [N]} \mathbb{E}\Big[\prod_{w_1 \in W_1} T_{d,N}(w_1 | \phi)\Big] \prod_{w_2 \in W_2} u_N(w_2 | \phi)$$

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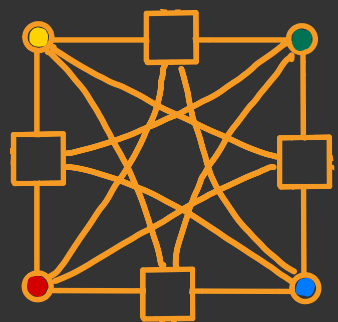
$G^\pi =$  , then using that $|u_N^{(i,j)}(k)| = O(1)$

$$\tau^0(G^\pi) = \frac{1}{N^{m/2+1}} \sum_{\phi : V^\pi \hookrightarrow [N]} \mathbb{E}\Big[\prod_{w_1 \in W_1} T_{d,N}(w_1|\phi)\Big] \prod_{w_2 \in W_2} u_N(w_2|\phi)$$

$$= O(N^{|\pi| - m/2 - 1}) = O(N^{4-2-1}) = O(N)$$

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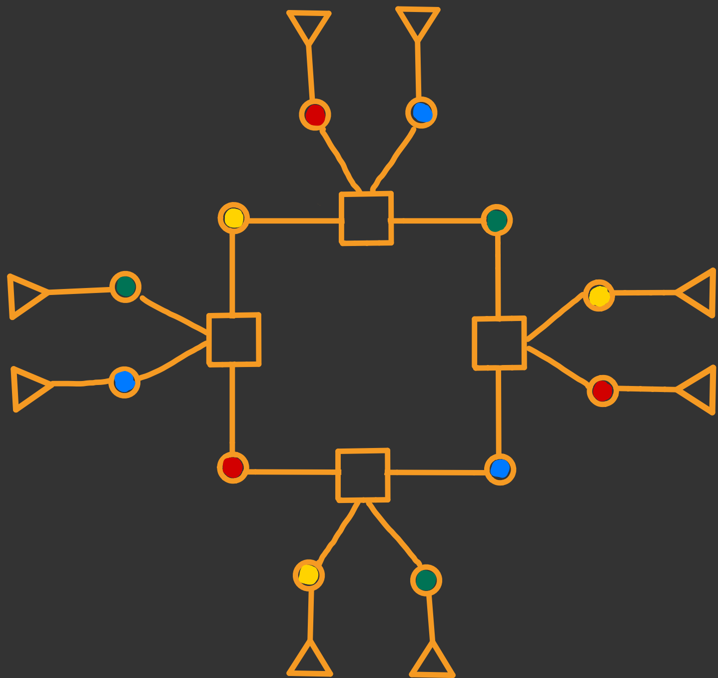
E.g. When the u_N's are canonical vectors or the vector $\left(\frac{1}{\sqrt{N}}, \dots, \frac{1}{\sqrt{N}}\right)$

E.g. When $\pi = \{B_1, B_2, B_3, B_4\}$ each block has two outer vertices, so by Cauchy-Schwarz



$$\tau^0(G^\pi) = \frac{1}{N^{m/2+1}} = O(N^{-3})$$

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Proof strategy

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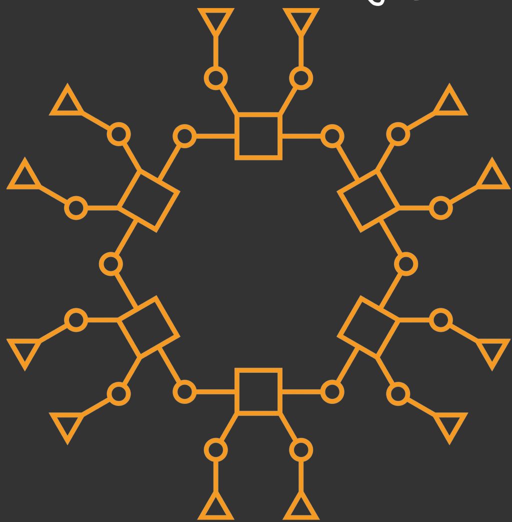
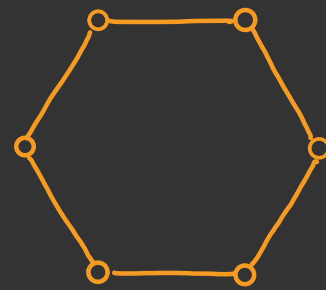
$G =$  $\longmapsto$ $H =$ 

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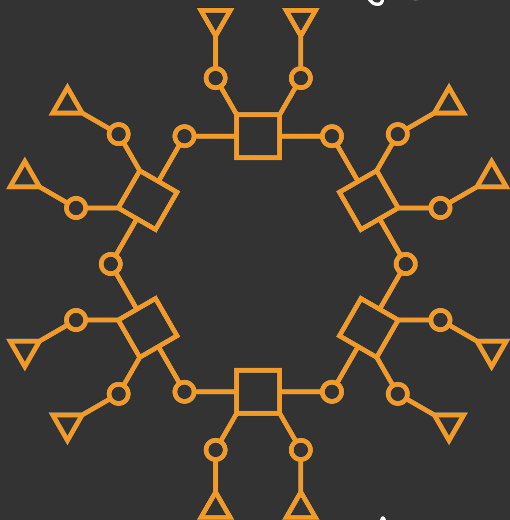
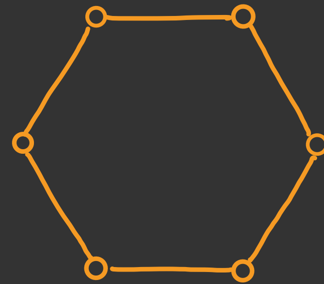
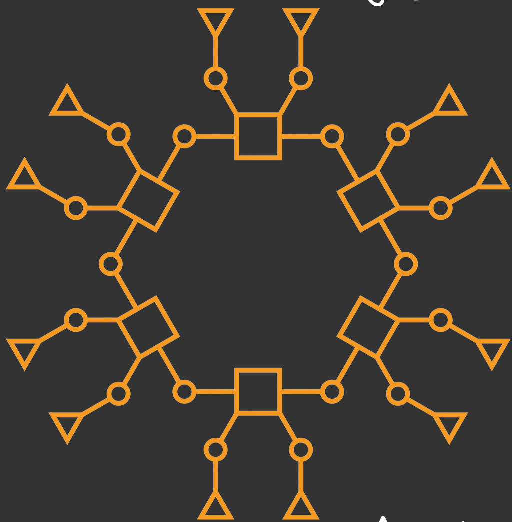
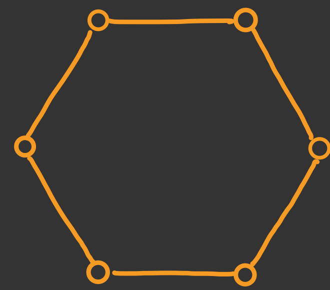
$G =$  $\longmapsto$ $H =$ 

And $G^{\pi} \longmapsto H^{\pi}$.

Proof strategy

$G =$  $\longmapsto$ $H =$ 

And $G^{\pi} \longmapsto H^{\pi}$.

- Intense combinatorics imply:

Lemma. $\lim\limits_{N \to \infty} \tau^{0}(G^{\pi}) = 0$ unless H^{π} is a double tree and $\pi \leq \{V_{in}, V_{out}\}$ (there are no inner-outer interactions).

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- Complication C: Computing $\lim_{N\to\infty} T^0(G^\pi)$ for the surviving terms is not easy.

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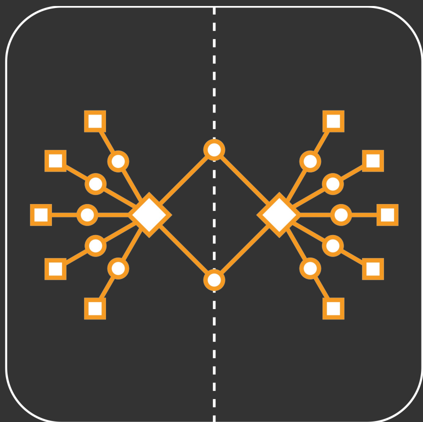
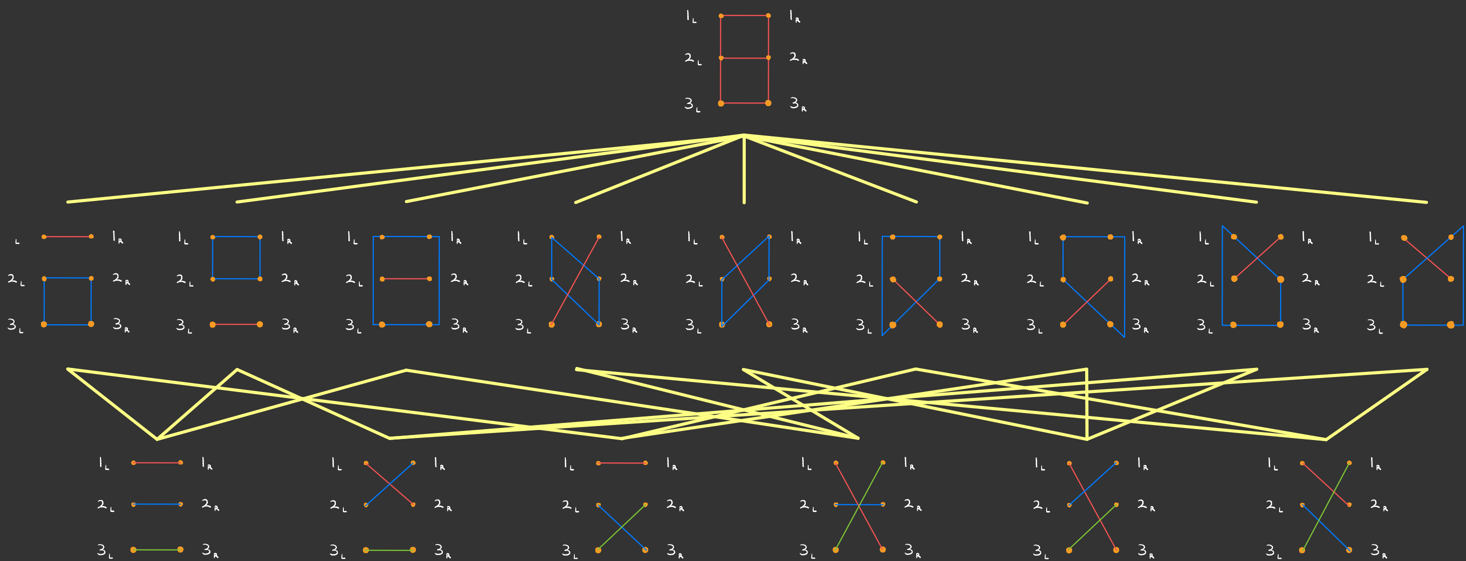
- Using the independence assumption the problem reduces to computing $\tau^\circ(G)$ for $m=2$

i.e. $\sum_{\pi \in P(V)} \sum_{\phi : V^\pi \hookrightarrow [N]}$ 

Uniform block permutations of $[d-2]_L \sqcup [d-2]_R$



- When the variance profile of $T_{d,N}$ is that of the GOTE we show that for $m=2$

$$T^0(G) = \frac{1}{d(d-1)} \left\langle u_N^{(i,1)} \odot \cdots \odot u_N^{(i,d-2)}, u_N^{(i',d-2)} \odot \cdots \odot u_N^{(i',d-2)} \right\rangle$$

Final comments

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Final comments

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- To show almost sure convergence we show concentration of

$$\mathrm{Tr}(G) = \frac{1}{N^{\frac{m}{2}+1}} \sum_{\phi: V \to [N]} \prod_{w_1 \in W_1} T_{d,N}(w_1 | \phi) \prod_{w_2 \in W_2} u_N(w_2 | \phi)$$

by bounding $\mathbb{E}\left[\left(\mathrm{Tr}(G) - \mathbb{E}\,\mathrm{Tr}(G) \right)^{2M} \right]$ for all M.

MERCI!